

Covariant Constitutive Relations and Relativistic Inhomogeneous Plasmas

J Gratus¹ and R W Tucker¹

Physics Department Lancaster University and the Cockcroft Institute

The notion of a two-point *susceptibility kernel* used to describe linear electromagnetic responses of dispersive continuous media in non-relativistic phenomena is generalized to accommodate the constraints required of a causal formulation in spacetimes with background gravitational fields. In particular the concepts of spatial material inhomogeneity and temporal non-stationarity are formulated within a fully covariant spacetime framework. This framework is illustrated by re-casting the Maxwell-Vlasov equations for a collisionless plasma in a form that exposes a 2-point electromagnetic susceptibility kernel in spacetime. This permits the establishment of a perturbative scheme for non-stationary inhomogeneous plasma configurations. Explicit formulae for the perturbed kernel are derived in both the presence and absence of gravitation using the general solution to the relativistic equations of motion of the plasma constituents. In the absence of gravitation this permits an analysis of collisionless damping in terms of a system of integral equations that reduce to standard Landau damping of Langmuir modes when the perturbation refers to a homogeneous stationary plasma configuration. It is concluded that constitutive modelling in terms of a 2-point susceptibility kernel in a covariant spacetime framework offers a natural extension of standard non-relativistic descriptions of simple media and that its use for describing linear responses of more general dispersive media has wide applicability in relativistic plasma modelling.

PACS numbers: 52.27.Ny, 41.20.-q, 52.25.Dg, 52.25.Fi, 52.25.Mq

I. INTRODUCTION

The behaviour of a material medium in response to electromagnetic and gravitational fields encompasses a vast range of classical and quantum physics. For media composed of a large collection of molecular or ionized structures recourse to a statistical description is required and this often leads to a coarser description in terms of a few thermodynamic variables and their correlations. Such a description relies on the efficacy of particular constitutive models or phenomenological constitutive data that serve to circumscribe its domain of applicability.

For phenomena where the relative motions of the constituents approach the speed of light in vacuo or the material experiences bulk accelerations or gravitational interactions such constitutive descriptions must be formulated within a relativistic framework. However even within a spacetime covariant formulation there remains great freedom in how to accommodate electromagnetic responses that depend on material dispersion induced by spatial correlations or temporal delays of electromagnetic interactions¹. The incorporation of such effects in a theoretical description often relies on a detailed structural model of the medium particularly if it is inhomogeneous or external gravitational gradients are relevant. Notwithstanding these complexities simple constitutive models have proved of considerable value for homogeneous polarizable media that exhibit temporal dispersion in a laboratory frame where gravity plays no essential role. Indeed the notion of permittivity and permeability tensors is often adequate to parametrize a large range of experimental linear responses of simple polarizable media to external static and dynamic electromagnetic fields. More generally, for non-dispersive media these tensors can be subsumed into a *susceptibility kernel* that readily accommodates special relativistic effects on the bulk motion of media.

In this article the degree to which the notion of a *susceptibility kernel* can be generalized to describe linear electromagnetic responses of dispersive continuous media is explored. In particular the effects of spatial material inhomogeneity and non-stationarity will be formulated within a fully covariant spacetime framework. In this manner the formulation can accommodate arbitrary gravitational and electromagnetic interactions. The framework will be illustrated by re-casting the Maxwell-Vlasov equations for a collisionless plasma in a form that exposes a 2-point² electromagnetic susceptibility kernel in an arbitrary external gravitational field. This permits the establishment of a perturbative scheme for non-stationary

inhomogeneous plasma configurations in terms of such a kernel. Explicit formulae for the perturbed kernel are derived in both the presence and absence of gravitation in terms of the general solution to the equations of motion of the plasma constituents. In the absence of gravitation this permits an analysis of collisionless damping in terms of a system of integral equations that reduce to standard Landau damping of Langmuir modes when the perturbation refers to a homogeneous stationary plasma configuration.

It is concluded that constitutive modelling in terms of a 2-point susceptibility kernel in a covariant spacetime framework offers a natural extension of standard non-relativistic descriptions of simple media and that its use for describing linear responses of more general dispersive media has wide applicability in relativistic plasma modelling.

II. CONSTITUTIVE RELATIONS

In the following spacetime M is considered a globally hyperbolic, topologically trivial four dimensional manifold endowed with a metric tensor g with signature $(-1, +1, +1, +1)$ describing gravitation. A closed 2-form F describes the electromagnetic field. The bundle of exterior p -forms over M is denoted $\Lambda^p M$ and its sections $\Gamma\Lambda^p M$ are p -forms on M . The bundle of all forms is $\Lambda M = \bigcup_{p=0}^{p=4} \Lambda^p M$. Associated with g is the Hodge map \star . Thus for $\alpha \in \Gamma\Lambda^p M$ its corresponding Hodge dual is denoted $\star\alpha \in \Gamma\Lambda^{4-p} M$. The tangent bundle over M is denoted TM and its sections ΓTM are vector fields on M . We call the 1-form $\tilde{J} = g(J, -) \in \Gamma\Lambda^1 M$ the *metric dual* of the vector field $J \in \Gamma TM$. Maxwell's equations for the electromagnetic field $F \in \Gamma\Lambda^2 M$ in a polarizable medium containing an electric current $J \in \Gamma TM$, satisfying the continuity (or current conservation) equation $d\star\tilde{J} = 0$, are written

$$dF = 0 \quad \text{and} \quad d\star G = -\star\tilde{J} \quad (1)$$

The excitation 2-form $G \in \Gamma\Lambda^2 M$ can always be expressed

$$G = \epsilon_0 F + \Pi \quad (2)$$

in terms of the permittivity ϵ_0 of free space. The polarization³ 2-form $\Pi \in \Gamma\Lambda^2 M$ results from all electromagnetic field sources not made explicit in J .

In general Π and J are non-linear functionals of F and other fields such as matter and initial data on any initial spacelike hypersurface $\Sigma_M \subset M$. Such functionals are the *constitutive relations* describing G and J in terms of F and these other fields.

It is convenient to introduce integration on a fibred manifold \mathcal{N} of dimension $n + r$ with projection $\pi_{\mathcal{N}} : \mathcal{N} \rightarrow N$ over a manifold N of dimension n . Thus at each point $\sigma \in N$ one has the fibre $\mathcal{N}_{\sigma} = \pi_{\mathcal{N}}^{-1}\{\sigma\} = \{(\sigma', \varsigma) \in \mathcal{N} \mid \pi_{\mathcal{N}}(\sigma', \varsigma) = \sigma\}$ so $\dim(\mathcal{N}_{\sigma}) = r$ is the fibre dimension. For $\alpha \in \Gamma\Lambda^{p+r}\mathcal{N}$ we define^{4,5} the form $\oint_{\pi_{\mathcal{N}}} \alpha \in \Gamma\Lambda^p N$ by

$$\int_N \beta \wedge \oint_{\pi_{\mathcal{N}}} \alpha = \int_{\mathcal{N}} \pi_{\mathcal{N}}^*(\beta) \wedge \alpha \quad (3)$$

for all $\beta \in \Gamma\Lambda^{n-p}N$.

In terms of local coordinates $(\sigma^1, \dots, \sigma^n)$ and $(\sigma^1, \dots, \sigma^n, \varsigma^1 \dots \varsigma^r)$ for patches on N and \mathcal{N} respectively, one may write the fibre integral

$$\left(\oint_{\pi_{\mathcal{N}}} \alpha \right) \Big|_{\sigma} = \sum_{1 \leq I_1 < \dots < I_p \leq n} d\sigma^{I_1} \wedge \dots \wedge d\sigma^{I_p} \int_{\varsigma \in \mathcal{N}_{\sigma}} i_{\partial/\partial\sigma^{I_p}} \dots i_{\partial/\partial\sigma^{I_1}} \alpha|_{(\sigma, \varsigma)} \quad (4)$$

where $\mathcal{N}_{\sigma} = \pi_{\mathcal{N}}^{-1}(\{\sigma\})$ is the fibre over the point $\sigma \in N$ and $i_{\partial/\partial\sigma^{I_k}}$ is the contraction on forms. Observe that if α does not contain the factor $d\varsigma^1 \wedge \dots \wedge d\varsigma^r$ then $\oint_{\pi_{\mathcal{N}}} \alpha = 0$. The proof of this is given in appendix lemma 2.

A key result of fibre integration, used to establish the current continuity equation, is that it commutes with the exterior derivative:

$$\left(d \oint_{\pi_{\mathcal{N}}} \alpha \right) \Big|_{\sigma} = \left(\oint_{\pi_{\mathcal{N}}} d\alpha \right) \Big|_{\sigma} \quad (5)$$

for σ not on the boundary of N provided the support of α does not intersect the boundary of \mathcal{N} . The proof is given in appendix lemma 3.

In general models for Π demand a knowledge of the dynamics of sources responsible for polarization as well as any permanent polarization that may exist in the medium. A full dynamical description depends on a specification of appropriate initial value data ζ on Σ_M . The exact structure of ζ depends on the sources of the polarization. For the plasma model described in section III the initial data corresponds to the velocity profile for each particle species at each point on Σ_M in the plasma.

In this article Π is considered to be an *affine* functional of F of the form

$$\Pi[F, \zeta] = \oint_{p_X} \chi \wedge p_Y^*(F) + Z[\zeta] \quad (6)$$

for some functional Z of ζ . The first term on the right is expressed in terms of the fibre integral of a two-point *susceptibility kernel* $\chi \in \Gamma\Lambda^4(M_X \times M_Y)$ expressible locally as

$$\chi = \frac{1}{4} \chi_{abcd}(x, y) dx^a \wedge dx^b \wedge dy^c \wedge dy^d \quad (7)$$

Here M_X and M_Y are two copies of M , locally coordinated by (x^0, \dots, x^3) and (y^0, \dots, y^3) respectively, with projections $p_X : M_X \times M_Y \rightarrow M_X$, $p_Y : M_X \times M_Y \rightarrow M_Y$, $p_X(x, y) = x$, $p_Y(x, y) = y$ and initial hypersurfaces $\Sigma_{M_X} \subset M_X$ and $\Sigma_{M_Y} \subset M_Y$. Throughout, summation is over Roman indices $a, b, c = 0, 1, 2, 3$ and Greek indices $\mu, \nu, \sigma = 1, 2, 3$.

To consistently remove any reference to M (without a subscript) let $F \in \Gamma\Lambda^2 M_Y$, $\epsilon_0 F \in \Gamma\Lambda^2 M_X$, $G \in \Gamma\Lambda^2 M_X$, $J \in \Gamma TM_X$ and $\Pi[F, \zeta] \in \Gamma\Lambda^2 M_X$. Thus ϵ_0 can be regarded as a map $\epsilon_0 : \Gamma\Lambda^2 M_Y \rightarrow \Gamma\Lambda^2 M_X$ which is the pullback of the natural isomorphism $M_X \rightarrow M_Y$, together with a scaling to accommodate the choice of electromagnetic units.

In terms of local coordinate bases on M_X and M_Y the components of (6) are

$$\Pi[F, \zeta]_{ab}(x) = \int_{y \in M} \frac{1}{4} \chi_{abcd}(x, y) F_{ef}(y) dy^{cdef} + Z[\zeta]_{ab} \quad (8)$$

in a multi-index notation with

$$dx^{a_1 \dots a_p} \equiv dx^{a_1} \wedge \dots \wedge dx^{a_p}$$

and

$$i_{a_1 \dots a_p}^{(x)} \equiv i_{\frac{\partial}{\partial x^{a_p}}} \dots i_{\frac{\partial}{\partial x^{a_1}}}$$

(Note the reverse order for internal contraction.) Summations over multi-indices $I \subset \{1, \dots, n\}$ considered as an ordered p -list $I_1 < I_2 < \dots < I_p$ of length $|I| = p$ will also be employed. Thus

$$dx^I \equiv dx^{I_1 \dots I_p} = dx^{I_1} \wedge \dots \wedge dx^{I_p}$$

and

$$i_I^{(x)} \equiv i_{I_1 \dots I_p}^{(x)} = i_{\frac{\partial}{\partial x^{I_p}}} \dots i_{\frac{\partial}{\partial x^{I_1}}}$$

so that, via summation, if $\alpha \in \Gamma\Lambda^p M$ then $dx^I \wedge i_I^{(x)} \alpha = \alpha$ where $|I| = p$.

In this notation the product manifold $M_X \times M_Y$ inherits the following maps that will be

employed below:

$$\begin{aligned}
d_X : \Gamma\Lambda^p(M_X \times M_Y) &\rightarrow \Gamma\Lambda^{p+1}(M_X \times M_Y), \\
d_X(\alpha) &= \frac{\partial\alpha_{IJ}}{\partial x^a} dx^a \wedge dx^I \wedge dy^J \\
d_Y : \Gamma\Lambda^p(M_X \times M_Y) &\rightarrow \Gamma\Lambda^{p+1}(M_X \times M_Y), \\
d_Y(\alpha) &= \frac{\partial\alpha_{IJ}}{\partial y^a} dy^a \wedge dx^I \wedge dy^J \\
\star_X : \Gamma\Lambda(M_X \times M_Y) &\rightarrow \Gamma\Lambda(M_X \times M_Y), \\
\star_X(\alpha) &= \alpha_{IJ}(\star dx^I) \wedge dy^J
\end{aligned}$$

where $\alpha = \alpha_{IJ} dx^I \wedge dy^J$

Since $F = dA$ and for A with compact support away from any boundary of M_Y it follows from (6) that

$$\Pi[F, \zeta] = - \int_{p_X} (d_Y \chi) \wedge p_Y^*(A) + Z[\zeta]$$

Hence $\Pi[F, \zeta]$ remains invariant⁶ under the gauge transformation

$$\chi \longrightarrow \chi + d_Y \check{\zeta} \tag{9}$$

for any $\check{\zeta} = \check{\zeta}_{abc} dx^{ab} \wedge dy^c \in \Gamma\Lambda^3(M_X \times M_Y)$. Since the support of A can be made arbitrarily small $d_Y \chi$ is uniquely specified by $\Pi[F, \zeta]$. Furthermore

$$d \star \Pi[F, \zeta] = - \int_{p_X} (d_X \star_X d_Y \chi) \wedge p_Y^*(A) + d \star Z[\zeta]$$

hence $d \star \Pi[F, \zeta]$ is invariant under the gauge transformation

$$\chi \longrightarrow \chi + d_Y \check{\zeta} + \star_X d_X \check{\xi} \tag{10}$$

for any $\check{\zeta} = \check{\zeta}_{abc} dx^{ab} \wedge dy^c$ and $\check{\xi} = \check{\xi}_{abc} dx^a \wedge dy^{bc}$. Similarly $d_X \star_X d_Y \chi$ is uniquely determined by $d \star \Pi[F, \zeta]$.

In general, the permittivity functional Π is a non-local functional in spacetime given by the integral (8). If χ is smooth, and not identically zero, then Π is always non-local. However for distributional susceptibility kernels it is possible for Π to remain local. In this category one has the local, linear Minkowski constitutive relations

$$\Pi[F] = \epsilon_0(\epsilon_r - 1) i_v F \wedge \tilde{v} + \epsilon_0(\mu_r^{-1} - 1) \star ((i_v \star F) \wedge F)$$

where $v \in \Gamma TM_Y$ is a vector field representing the bulk 4-velocity of the medium and $\epsilon_r, \mu_r \in \Gamma \Lambda^0 M_Y$ are the relative permittivity and permeability scalars of the medium. These relations can be represented by a distributional susceptibility kernel with support on the diagonal set $\{(x, y) \in M_X \times M_Y | x = y\}$.

In general Π is said to be causal on all of M if $\Pi|_x$ only depends of the values of F which lie on or within the past light-cone^{7,8} $J^-(x) \subset M_Y$ of x . If Π depends on ζ it may be causal on M_X^+ where $M_X^+ = \Sigma_{M_X} \cup \{x \text{ lies to the future of } \Sigma_{M_X}\}$. The functional Π is causal on M_X^+ if $\Pi[F, \zeta]|_x$ only depends on the values of F and ζ which lie on or within its past light-cone $J^-(x) \cap M_X^+$ of x and $x \in M_X^+$. The data functional Z is casual on M_X^+ if $Z[\zeta]|_x$ depends only on $\zeta \in \Sigma_{M_X} \cap J^-(x)$ for all $x \in M_X^+$. For Π to be causal on M_X^+ it is necessary and sufficient (lemma 5 in the appendix) that the following be satisfied:

- Z is causal on M_X^+ ,
- $(d_Y \chi)|_{(x,y)} = 0$ for all $(x, y) \in M_X^+ \times M_Y^+$ such that $y \notin J^-(x)$ and
- $\iota_{\Sigma_{M_Y}}^*(\chi)|_{(x,y)} = 0$ for all $(x, y) \in M_X^+ \times \Sigma_{M_Y}$ such that $y \notin J^-(x)$, where $\iota_{\Sigma_{M_Y}} : M_X^+ \times \Sigma_{M_Y} \hookrightarrow M_X^+ \times M_Y^+$ is the natural embedding.

A. Spacetime homogeneous constitutive relations for media in Minkowski spacetime

Minkowski spacetime has properties that underpin the notions of material spatial homogeneity and stationary processes. Being isomorphic to a real 4-dimensional vector space it can be given an affine structure in addition to its light-cone structure. Physically this implies that no particular point in a spacetime without gravitation has a distinguished status and the concepts of material and field energy, momentum and angular momentum can be defined in terms of the Killing symmetries of the spacetime metric. Since all points of the spacetime are equivalent relative to this affine structure it is sufficient to denote M_X and M_Y by M and, relative to any point chosen as origin, a point with coordinates x can be identified with a vector denoted by $x \in \mathbb{R}^4$. It is then convenient to introduce the Minkowski translation map $A_z : M \rightarrow M$, $A_z(x) = x + z$ that maps points x to $x + z$ on M .

If the electromagnetic properties of an unbounded medium are independent of location in spacetime they will be called *spacetime homogeneous*. Such electromagnetic constitutive

properties imply that variations in F at event $y \in M$ produce an induced variation in a functional $\Pi_{\mathbf{H}}[F]$ at event $x \in M$, via a kernel $\chi_{abcd}(x, y)$ that depends on the 4-vector $x - y$. If the constitutive relation is causal then there is no induced variation if $x \notin J^+(y)$. Furthermore in a spacetime homogeneous medium $Z[\zeta] = Z_{\mathbf{H}}$ where $Z_{\mathbf{H}} \in \Gamma\Lambda^2 M$ is independent of ζ .

In terms of A_z an electromagnetic constitutive functional $\Pi_{\mathbf{H}}$ is given by

$$\Pi_{\mathbf{H}}[F] = \oint_{p_x} \chi \wedge p_Y^*(F) + Z_{\mathbf{H}} \quad (11)$$

The functional $\Pi_{\mathbf{H}}$ is said to be *spacetime homogeneous*⁹ if

$$\Pi_{\mathbf{H}}[A_z^* F] = A_z^* \Pi_{\mathbf{H}}[F] \quad (12)$$

This follows if the susceptibility kernel χ satisfies

$$\chi|_{(x+z, y+z)} = \chi|_{(x, y)} \quad (13)$$

and $A_z^* Z_{\mathbf{H}} = Z_{\mathbf{H}}$. The contribution $Z_{\mathbf{H}}$ may model the presence of an externally prescribed stationary uniform permanent magnetic or electric polarization. Equation (13) implies the components of χ in (7) can be written

$$\chi_{abcd}(x, y) = X_{abcd}(x - y) \quad (14)$$

where

$$X_{abcd}(x) = \chi_{abcd}(x, 0) \quad (15)$$

Thus, in a Minkowski spacetime for materials with electromagnetic spacetime homogeneous properties, (8) can be written in terms of a convolution integral:

$$\begin{aligned} \Pi_{\mathbf{H}}[F]_{ab}(x) &= \frac{1}{4} \int_{y \in M} X_{abcd}(x - y) F_{ef}(y) dy^{cdef} + (Z_{\mathbf{H}})_{ab} \\ &\equiv \frac{1}{4} \epsilon^{cdef} (X_{abcd} * F_{ef})(x) + (Z_{\mathbf{H}})_{ab} \end{aligned} \quad (16)$$

where $\epsilon^{cdef} = \pm 1, 0$ denotes the Levi-Civita alternating symbol in coordinates in which the metric tensor takes the form $g = \eta_{ab} dx^a \otimes dx^b$ where $\eta_{ab} = \text{diag}(-1, +1, +1, +1)$. In these coordinates the $(Z_{\mathbf{H}})_{ab}$ are all constants.

Let $\hat{F}_{ef}(k)$ and $\hat{\Pi}_{\mathbf{H}}[F]_{ab}(k)$ denote the Fourier transforms of $F_{ef}(x)$ and $\Pi_{\mathbf{H}}[F]_{ab}(x)$ respectively, i.e.

$$\hat{F}_{ef}(k) = \int_{x \in \mathbb{R}^4} F_{ef}(x) e^{ik \cdot x} dx^{0123}$$

and

$$\hat{\Pi}_{\mathbf{H}}[F]_{ab}(k) = \int_{x \in \mathbb{R}^4} \Pi_{\mathbf{H}}[F]_{ab}(x) e^{ik \cdot x} dx^{0123}$$

where $k = k_a dx^a$, $k \cdot x = k_a x^a$. Similarly let $\hat{X}_{ab}{}^{ef}(k)$ be the Fourier transformation of $\frac{1}{2} \epsilon^{cdef} X_{abcd}(x)$, i.e.

$$\hat{X}_{ab}{}^{ef}(k) = \frac{1}{2} \epsilon^{cdef} \int_{x \in \mathbb{R}^4} X_{abcd}(x) e^{ik \cdot x} dx^{0123} \quad (17)$$

If $Z_{\mathbf{H}} = 0$ then it follows from (16) that:

$$\hat{\Pi}_{\mathbf{H}}[F]_{ab}(k) = \frac{1}{2} \hat{X}_{ab}{}^{cd}(k) \hat{F}_{cd}(k) \quad (18)$$

Since χ_{abcd} is a real function on M its Fourier transform satisfies

$$\hat{X}_{ab}{}^{cd}(k)^* = \hat{X}_{ab}{}^{cd}(-k)$$

The 36 components of $\hat{X}_{ab}{}^{cd}(k)$ subject to this symmetry can be expressed in terms of permittivity, permeability and magneto-electric tensors relative to any observer frame. A specification of these components together with relations that determine the electric current J serve as an electromagnetic model for a spacetime homogeneous medium in Minkowski spacetime. If the medium lacks this electromagnetic homogeneity recourse to the Fourier transform (16) is not possible and the constitutive properties must be given in terms of a 2-point kernel and (8).

III. CONSTITUTIVE MODELS FOR A COLLISIONLESS IONIZED PLASMA

As noted in the introduction the computation of the susceptibility for homogeneous stationary dispersive media owes much to phenomenological models and input from experiment. For certain conductors, semi-conductors, insulators and low-dimensional structures much can also be learnt from the application of quantum theory. For inhomogeneous and anisotropic

media subject to non-stationary electromagnetic fields linear responses are often the subject of a perturbation approach. This is particularly so in the case of ionized gases.

As an application of the above formalism the classical linear response of a fully ionized inhomogeneous non-stationary collisionless plasma to a perturbation is considered in the presence of an arbitrary background gravitational field. The perturbed constitutive tensor will be calculated in terms of solutions to the classical Maxwell-Vlasov equations for the system. This system is described in terms of the electromagnetic 2-form $F \in \Gamma\Lambda^2 M^+$ over a gravitational spacetime M^+ , lying in the future of an initial hypersurface Σ_M , and a collection of one-particle “distribution” forms (of degree 6), $\theta^{[\alpha]} \in \Gamma\Lambda^6 \mathcal{E}^+$ (one for each charged species of particle $[\alpha]$ with mass $m^{[\alpha]}$ and charge $q^{[\alpha]}$) on the upper unit hyperboloid bundle $\pi : \mathcal{E}^+ \rightarrow M^+$ over M^+ . The 7-dimensional manifold \mathcal{E}^+ is a sub-bundle of the 8-dimensional tangent bundle TM^+ over M^+ whose sections are all future pointing time-like unit vector fields on M^+ . Thus generic elements of \mathcal{E}^+ can be written (z, w) with $z \in M^+$, $\pi(z, w) = z$ and $g(w, w) = -1$. The initial values of the one-particle forms are given on the hypersurface $\Sigma_{\mathcal{E}}$ where $\Sigma_{\mathcal{E}} = \pi^{-1}\{\Sigma_M\} \subset \mathcal{E}^+$.

The Maxwell-Vlasov system is usually written in terms of the Maxwell system in vacuo and *all* sources are contained in the total current $J \in \Gamma TM^+$. This in turn is given by the sum over each species current

$$J = \sum_{[\alpha]} J^{[\alpha]} \quad (19)$$

where $J^{[\alpha]} \in \Gamma TM^+$. Thus in terms of F and J the Maxwell subsystem is

$$dF = 0 \quad \text{and} \quad \epsilon_0 d \star F = - \star \tilde{J} \quad (20)$$

The dynamic equations for each $\theta^{[\alpha]}$ can be written succinctly in terms of forms on \mathcal{E}^+ and a collection of Liouville vector fields $W^{[\alpha]} \in \Gamma T\mathcal{E}^+$ describing the flow of the charged particles associated with each species $[\alpha]$:

$$W^{[\alpha]}|_{(z,w)} = \mathcal{H}_{(z,w)}(z, w) + \frac{q^{[\alpha]}}{m^{[\alpha]}} \mathcal{V}_{(z,w)}(\widetilde{i_{(z,w)} F}) \quad (21)$$

in terms of certain horizontal and vertical lifts¹⁰. With these vector fields the distribution forms $\theta^{[\alpha]}$ are defined to satisfy the collisionless conditions:

$$d\theta^{[\alpha]} = 0 \quad (22)$$

and

$$i_{W^{[\alpha]}} \theta^{[\alpha]} = 0 \quad (23)$$

To close this system one requires:

$$\star \widetilde{J}^{[\alpha]} = q^{[\alpha]} \oint_{\pi} \theta^{[\alpha]} \quad (24)$$

The closure of $\theta^{[\alpha]}$ leads, from (5), to the continuity equation for each species current:

$$d \star \widetilde{J}^{[\alpha]} = d \left(\oint_{\pi} \theta^{[\alpha]} \right) = \oint_{\pi} d\theta^{[\alpha]} = 0 \quad (25)$$

so the total current 3-form $\star \widetilde{J}$ is closed away from the boundary Σ_M .

A local coordinate system (z^0, \dots, z^3) for a region containing z on M^+ induces a local coordinate system $(z^0, \dots, z^3, w^1, w^2, w^3)$ on \mathcal{E}^+ . Since $\mathcal{E}^+ \subset TM^+$ the tangent vector for a generic element $(z, w) \in \mathcal{E}^+$ may be written

$$(z, w) = w^a \frac{\partial}{\partial z^a} \Big|_z \in \mathcal{E}_z^+ \subset T_z M^+$$

where $\mathcal{E}_z^+ = \pi^{-1}(\{z\})$ is the 3-dimensional fibre of \mathcal{E}^+ over z coordinated by (w^1, w^2, w^3) and $w^0(z, w)$ is the solution to $g_{ab} w^a w^b = -1$ with $w^0 > 0$. All indices in the range $0, 1, 2, 3$ are raised and lowered using g^{ab} and g_{ab} so that $w_0 = w^a g_{a0}$. Given a pair of vectors $(z, w), (z, v) \in \mathcal{E}_z^+ \subset T_z M^+$ the horizontal lift of the vector (z, v) to the point $(z, w) \in \mathcal{E}^+$ will be denoted $\mathcal{H}_{(z,w)}(z, v) \in T_{(z,w)} \mathcal{E}^+$ and is given by

$$\mathcal{H}_{(z,w)}(z, v) = \left(v^a \frac{\partial}{\partial z^a} - \Gamma^\nu_{ef}(z) w^e v^f \frac{\partial}{\partial w^\nu} \right) \Big|_{(z,w)} \quad (26)$$

where Γ^a_{ef} are the Christoffel symbols determined by the metric components g^{ab} . Furthermore if $g(v, w) = 0$ then the vertical lift of the vector (z, v) to the point $(z, w) \in \mathcal{E}^+$ is given by

$$\mathcal{V}_{(z,w)}(z, v) = \left(v^\mu \frac{\partial}{\partial w^\mu} \right) \Big|_{(z,w)} \in T_{(z,w)} \mathcal{E}^+ \quad (27)$$

Thus from (21), each Liouville vector field in these coordinates can be expressed as

$$W^{[\alpha]}|_{(z,w)} = w^a \frac{\partial}{\partial z^a} + \left(-\Gamma^\nu_{ef}(z) w^e w^f + \frac{q^{[\alpha]}}{m^{[\alpha]}} F_{ef}(z) g^{\nu e} w^f \right) \frac{\partial}{\partial w^\nu} \quad (28)$$

Denote by $\Omega \in \Gamma \Lambda^7 \mathcal{E}^+$ the natural 7-form measure on \mathcal{E}^+ given in these coordinates by

$$\Omega = \frac{|\det g|}{w_0} dz^{0123} \wedge dw^{123} \quad (29)$$

In ref. 11, eqn. (94) it is shown that for all species $[\alpha]$

$$di_{W^{[\alpha]}}\Omega = 0 \quad (30)$$

The *distribution function* $f^{[\alpha]} \in \Gamma\Lambda^0\mathcal{E}^+$ relative to Ω for the species $[\alpha]$ is defined implicitly via

$$\theta^{[\alpha]} = i_{W^{[\alpha]}}(f^{[\alpha]}\Omega) \quad (31)$$

From (30, 31) it follows that (23) is equivalent to

$$W^{[\alpha]}(f^{[\alpha]}) = 0, \quad (32)$$

and from (24) the components of the species current $[\alpha]$ are given in terms of $f^{[\alpha]}(z, w)$ by

$$J^{[\alpha]b}(z) = q^{[\alpha]} \int_{\mathcal{E}_z^+} \frac{w^b |(\det g)(z)|^{1/2}}{w_0(z, w)} f^{[\alpha]}(z, w) dw^{123} \quad (33)$$

A. Perturbation analysis

Let $\theta_1^{[\alpha]} \in \Gamma\Lambda^6\mathcal{E}^+$ and $F_1 \in \Gamma\Lambda^2M^+$ be perturbations of $\theta_0^{[\alpha]}$ and F_0 , i.e.

$$\theta^{[\alpha]} = \theta_0^{[\alpha]} + \theta_1^{[\alpha]} + \dots \quad \text{and} \quad F = F_0 + F_1 + \dots \quad (34)$$

where

$$\begin{aligned} d\theta_0^{[\alpha]} &= 0, & i_{W_0^{[\alpha]}}\theta_0^{[\alpha]} &= 0, \\ dF_0 &= 0, & \epsilon_0 d \star F_0 &= - \sum_{[\alpha]} q^{[\alpha]} \oint_{\pi} \theta_0^{[\alpha]} \end{aligned} \quad (35)$$

and

$$W_0^{[\alpha]}|_{(z,w)} = \mathcal{H}_{(z,w)}(z, w) + \frac{q^{[\alpha]}}{m^{[\alpha]}} \mathcal{V}_{(z,w)}(\widetilde{i_{(z,w)}F_0}) \quad (36)$$

i.e. given by substituting $F = F_0$ into (28). Substituting F into (21) yields $W^{[\alpha]} = W_0^{[\alpha]} + W_1^{[\alpha]} + \dots$ where $W_1^{[\alpha]} = \hat{W}_1^{[\alpha]}(F_1)$ and the map $\hat{W}_1 : \Gamma\Lambda^2M^+ \rightarrow \Gamma T\mathcal{E}^+$ is given by

$$\hat{W}_1^{[\alpha]}(F_1)|_{(z,w)} = \frac{q^{[\alpha]}}{m^{[\alpha]}} \mathcal{V}_{(z,w)}(\widetilde{i_{(z,w)}F_1}) \quad (37)$$

The first order linear system for the perturbation (θ_1, F_1) is then

$$d\theta_1^{[\alpha]} = 0, \quad (38)$$

$$i_{W_0^{[\alpha]}}\theta_1^{[\alpha]} = -i_{\hat{W}_1^{[\alpha]}(F_1)}\theta_0^{[\alpha]}, \quad (39)$$

$$dF_1 = 0, \quad (40)$$

$$\epsilon_0 d \star F_1 = - \sum_{[\alpha]} q^{[\alpha]} \oint_{\pi} \theta_1^{[\alpha]} \quad (41)$$

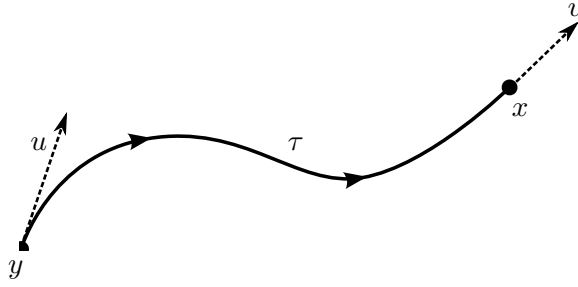


FIG. 1. A segment of the solution curve $C_{(x,v)}$ to the unperturbed Lorentz force equation (46) with final position x , final velocity (x, v) , initial position $y = C_{(x,v)}(\tau)$ and initial velocity $(y, u) = \dot{C}_{(x,v)}(\tau)$.

Using (5) and (38) it follows that each species current in the sum on the right hand side of (41) is closed away from the initial hypersurface Σ_M . In terms of the excitation field $G_1 \in \Gamma\Lambda^2 M^+$ equation (41) will be written

$$d \star G_1 = 0 \quad (42)$$

where

$$G_1 = \epsilon_0 F_1 + \Pi_1[F_1, \zeta_1] \quad (43)$$

for some linear functional Π_1 of F_1 and ζ such that

$$d \star \Pi_1[F_1, \zeta_1] = - \sum_{[\alpha]} \oint_{\pi} \theta_1^{[\alpha]} \quad (44)$$

and $\zeta_1 = \{\zeta_1^{[\alpha_1]}, \zeta_1^{[\alpha_2]}, \dots\}$ where $\zeta_1^{[\alpha]} = \xi_1^{[\alpha]}|_{\Sigma_{\mathcal{E}_Y}}$ for some $\xi_1^{[\alpha]} \in \Gamma\Lambda^5 \mathcal{E}_Y^+$ which solves $\theta_1^{[\alpha]} = d\xi_1^{[\alpha]}$. Thus $\zeta_1^{[\alpha]}$ is related to the initial velocity profile of the species $[\alpha]$.

In the next section III B the general susceptibility kernel $\chi \in \Gamma\Lambda^0(M_X^+ \times M_Y^+)$ and linear functional Z_1 , determined by $\theta_0^{[\alpha]}$ and F_0 , are found such that

$$\Pi_1[F_1, \zeta_1]|_x = \oint_{p_X} \chi \wedge p_Y^*(F_1) + Z_1[\zeta_1] \quad (45)$$

satisfies (44).

B. A general formula for the functional Π_1 in an unbounded plasma

In this section a general expression for a susceptibility kernel will be constructed in terms of the integral curves of the vector field $W_0^{[\alpha]} \in \Gamma T\mathcal{E}^+$. Such curves describe segments of

particle world lines under the influence of the Lorentz force due to the external electromagnetic field F_0 . Although, for a general F_0 , it is not possible to derive an analytic form for such integral curves, special cases are amenable to an analytic analysis.

It proves convenient to let the final and initial states of each species of particle reside in fibres over M_X^+ and M_Y^+ respectively, bounded by the equivalent hypersurfaces $\Sigma_{M_X} \subset M_X^+$ and $\Sigma_{M_Y} \subset M_Y^+$. Thus the corresponding upper unit hyperboloid bundles $\pi_X : \mathcal{E}_X^+ \rightarrow M_X^+$ and $\pi_Y : \mathcal{E}_Y^+ \rightarrow M_Y^+$ with boundary hypersurfaces $\Sigma_{\mathcal{E}_X} \subset \mathcal{E}_X^+$ and $\Sigma_{\mathcal{E}_Y} \subset \mathcal{E}_Y^+$ are used to accommodate the final and initial 4-velocities of the particles. The generic elements of these bundles are written $(x, v) \in \mathcal{E}_X^+$ and $(y, u) \in \mathcal{E}_Y^+$ where $x \in M_X^+$, $y \in M_Y^+$ and $g(v, v) = g(u, u) = -1$. The induced coordinate systems for \mathcal{E}_X^+ and \mathcal{E}_Y^+ are $(x^0, \dots, x^3, v^1, v^2, v^3)$ and $(y^0, \dots, y^3, u^1, u^2, u^3)$. Let $v^0(x, v)$, $v_0(x, v)$, $u^0(y, u)$ and $u_0(y, u)$ be defined in the same way as $w^0(z, w)$ and $w_0(z, w)$.

The contribution to the tensor $\Pi_1[F_1, \zeta_1]$ due to all dynamic sources, arises from all particle histories in the past light cone of $x \in M_X^+$. The history of the species particle $[\alpha]$ which passes through event x with 4-velocity $(x, v) \in \mathcal{E}_X^+$ will therefore be parametrized by negative proper time τ : $C_{(x,v)}^{[\alpha]} : [\tau_0^{[\alpha]}(x, v), 0] \rightarrow M^+$, $\tau \mapsto C_{(x,v)}^{[\alpha]}(\tau)$. Such a history is the unique solution to the Lorentz force equation

$$\nabla_{\dot{C}_{(x,v)}^{[\alpha]}} \dot{C}_{(x,v)}^{[\alpha]} = \frac{q^{[\alpha]}}{m^{[\alpha]}} (\widetilde{i_{\dot{C}_{(x,v)}^{[\alpha]}} F_0}) \quad (46)$$

with

$$g(\dot{C}_{(x,v)}^{[\alpha]}, \dot{C}_{(x,v)}^{[\alpha]}) = -1 \quad (47)$$

and final condition

$$C_{(x,v)}^{[\alpha]}(0) = x, \quad \dot{C}_{(x,v)}^{[\alpha]}(0) = (x, v) \quad (48)$$

where $\dot{C}_{(x,v)}^{[\alpha]}(\tau) = C_{(x,v)\star}^{[\alpha]}(\partial_\tau|_\tau) = \dot{C}_{(x,v)}^{[\alpha]a}(\tau) \frac{\partial}{\partial x^a}$ and the value $\tau_0^{[\alpha]}(x, v) \leq 0$ solves

$$C_{(x,v)}^{[\alpha]}(\tau_0^{[\alpha]}(x, v)) \in \Sigma_{M_Y} \quad (49)$$

This defines the prolongation of C , $\dot{C}_{(x,v)}^{[\alpha]} : [\tau_0^{[\alpha]}(x, v), 0] \rightarrow \mathcal{E}^+$. For each species $[\alpha]$, $(x, v) \in \mathcal{E}_X^+$ and $\tau \in [\tau_0^{[\alpha]}(x, v), 0]$ let $(y, u) \in \mathcal{E}_Y^+$ denote the initial state, i.e. $y = C_{(x,v)}^{[\alpha]}(\tau)$ and $(y, u) = \dot{C}_{(x,v)}^{[\alpha]}(\tau)$, see figure 1.

The *family* of *all* such histories is described in terms of the maps

$$\phi^{[\alpha]} : \mathcal{N}_X^{[\alpha]} \rightarrow \mathcal{E}_Y^+, \quad \phi^{[\alpha]}(\tau, x, v) = \dot{C}_{(x,v)}^{[\alpha]}(\tau) \quad (50)$$

where

$$\mathcal{N}_X^{[\alpha]} = \{(\tau, x, v) \in \mathbb{R}^- \times \mathcal{E}_X^+ \mid \tau_0^{[\alpha]}(x, v) \leq \tau \leq 0\}$$

The manifold $\mathcal{N}_X^{[\alpha]}$ with boundary is naturally a fibre bundle over \mathcal{E}_X^+ with projection $\varpi_X^{[\alpha]} : \mathcal{N}_X^{[\alpha]} \rightarrow \mathcal{E}_X^+$, $(\tau, x, v) \mapsto \varpi_X^{[\alpha]}(\tau, x, v) = (x, v)$ and for any form $\alpha \in \Gamma\Lambda^p \mathcal{N}_X$ it follows from (4) that

$$\int_{\varpi_X^{[\alpha]}} \alpha = dx^I \wedge dy^J \int_{\tau_0^{[\alpha]}(x, v)}^0 \alpha^{(1)}(\tau, x, v) d\tau$$

where $\alpha = \alpha^{(1)}(\tau, x, v) dx^I \wedge dy^J \wedge d\tau + \alpha^{(2)}(\tau, x, v) dx^I \wedge dy^J$.

Let $\Gamma\Lambda_{\Sigma_{\mathcal{E}_Y}}^5 \mathcal{E}_Y^+$ be the set of sections over $\Sigma_{\mathcal{E}_Y}$ with values in $\Lambda^5 \mathcal{E}_Y^+$, i.e. if $\alpha \in \Gamma\Lambda_{\Sigma_{\mathcal{E}_Y}}^5 \mathcal{E}_Y^+$ then for each $(y, u) \in \Sigma_{\mathcal{E}_Y}$, $\alpha|_{(y, u)} \in \Lambda_{(y, u)}^5 \mathcal{E}_Y^+$. Let the map $\varphi^{[\alpha]} : \Gamma\Lambda_{\Sigma_{\mathcal{E}_Y}}^5 \mathcal{E}_Y^+ \rightarrow \Gamma\Lambda^5 \mathcal{E}_X^+$ be given by

$$\varphi^{[\alpha]}(\alpha)|_{(x, v)} = \phi_{\tau_0^{[\alpha]}(x, v)}^{[\alpha]\star}(\alpha|_{\tau_0^{[\alpha]}(x, v)}) \in \Lambda_{(x, v)}^5 \mathcal{E}_X^+ \quad (51)$$

where $\phi_\tau^{[\alpha]} : \mathcal{E}_X^+ \rightarrow \mathcal{E}^+$, $\phi_\tau^{[\alpha]}(x, v) = \phi(\tau, x, v)$. For each species $[\alpha]$ let the initial data be given by $\zeta_1^{[\alpha]} \in \Gamma\Lambda_{\Sigma_{\mathcal{E}_Y}}^5 \mathcal{E}_Y^+$ with $i_{W_0^{[\alpha]}} \zeta_1^{[\alpha]} = 0$.

In terms of these maps, it will now be shown that the general polarization functional Π_1 on M_X^+ is given by

$$\begin{aligned} \Pi_1[F_1, \zeta_1] &= \sum_{[\alpha]} q^{[\alpha]} \star \int_{\pi_X} \int_{\varpi_X^{[\alpha]}} d\tau \wedge \phi^{[\alpha]\star}(i_{\tilde{W}_1^{[\alpha]}(F_1)} \theta_0^{[\alpha]}) + \star d(\Xi_1[F_1]) \\ &\quad + \sum_{[\alpha]} q^{[\alpha]} \star \int_{\pi_X} \varphi^{[\alpha]}(\zeta_1^{[\alpha]}) + \star d(\tilde{Z}_1[\zeta_1]) \end{aligned} \quad (52)$$

where Ξ_1 and \tilde{Z}_1 are arbitrary linear functionals of F_1 and ζ_1 respectively. The excitation $\Pi_1[F_1, \zeta_1]$, in (52), is the general solution to (44) where the source θ_1 satisfies (38,39). The first two terms on the right hand side of (52) are linear functionals of F_1 whereas the last term is a linear functional of the initial data ζ_1 . Clearly $\star d(\Xi_1[F_1])$ and $\star d(\tilde{Z}_1[\zeta_1])$ are in the kernel of $d\star$, the homogeneous differential operator associated with (44).

The proof that (52) solves (44) requires the following lemma which is proved in the appendix.

Lemma 1. *Let N be a manifold with a boundary $\Sigma_N \subset N$ and let $V \in \Gamma TN$ be a non-vanishing vector field on N such that every integral curve of V intersects Σ_N precisely once.*

For each $\sigma \in N$ let the integral curve of V terminating at σ be given by $\gamma_\sigma : [\tau_0(\sigma), 0] \rightarrow N$ where $\gamma_\sigma(0) = \sigma$ and $\gamma_\sigma(\tau_0(\sigma)) \in \Sigma_N$. The set $\mathcal{N} = \{(\sigma, \tau) \in \mathbb{R}^- \times N \mid \tau_{\min}(\sigma) \leq \tau \leq 0\}$ is a fibred manifold over N with projection $\varpi_N : \mathcal{N} \rightarrow N$, $(\tau, \sigma) \mapsto \varpi_N(\tau, \sigma) = \sigma$. The family of integral curves of V can be described by the map $\phi_N : \mathcal{N} \rightarrow N$, $\phi_N(\tau, \sigma) = \gamma_\sigma(\tau)$. Let $\zeta \in \Gamma\Lambda_{\Sigma_N}^p N$ such that $i_V \zeta = 0$, i.e. ζ is a p -form on Σ_N with values in $\Lambda^p N$. Let $\varphi_N : \Gamma\Lambda_{\Sigma_N}^p N \rightarrow \Gamma\Lambda^p N$ be given by $\varphi_N(\zeta)|_\sigma = \phi_{N, \tau_0(\sigma)}^*(\zeta|_{\tau_0(\sigma)}) \in \Lambda_{\Sigma_N}^p N$.

If $\beta \in \Gamma\Lambda^p N$ is a p -form on N with compact support such that $i_V \beta = 0$ and $\xi \in \Gamma\Lambda^p N$ has the form

$$\xi = \int_{\varpi_N} \phi_N^*(\beta) \wedge d\tau + \varphi_N(\zeta) \quad (53)$$

then

$$i_V d\xi = \beta \quad (54)$$

and $\xi|_{\Sigma_N} = \zeta$.

This lemma is applied with $N = \mathcal{E}_X^+$, $\varpi_N = \varpi_X^{[\alpha]}$, $V = W_0^{[\alpha]}$, $\tau_0 = \tau_0^{[\alpha]}$, $\phi_N = \phi^{[\alpha]}$, $\varphi_N = \varphi^{[\alpha]}$, $\zeta = \zeta_1^{[\alpha]}$ and

$$\beta = -i_{\hat{W}_1^{[\alpha]}(F_1)} \theta_0^{[\alpha]} \quad (55)$$

Thus ξ in (53) becomes the 5-form $\xi_1^{[\alpha]} \in \Gamma\Lambda^5 \mathcal{E}_X^+$,

$$\xi_1^{[\alpha]} = - \int_{\varpi_X^{[\alpha]}} \phi^{[\alpha]*} (i_{\hat{W}_1^{[\alpha]}(F_1)} \theta_0^{[\alpha]}) \wedge d\tau + \varphi^{[\alpha]}(\zeta_1^{[\alpha]}) = \int_{\varpi_X^{[\alpha]}} d\tau \wedge \phi^{[\alpha]*} (i_{\hat{W}_1^{[\alpha]}(F_1)} \theta_0^{[\alpha]}) + \varphi^{[\alpha]}(\zeta_1^{[\alpha]}) \quad (56)$$

since $\deg(\phi^{[\alpha]*}(i_{\hat{W}_1^{[\alpha]}(F_1)} \theta_0^{[\alpha]})) = 5$. In order to satisfy (38) let

$$\theta_1^{[\alpha]} = d\xi_1^{[\alpha]} \quad (57)$$

Furthermore from (54) and (55)

$$i_{W_0^{[\alpha]}} \theta_1^{[\alpha]} = i_{W_0^{[\alpha]}} d\xi_1^{[\alpha]} = -i_{\hat{W}_1^{[\alpha]}(F_1)} \theta_0^{[\alpha]}$$

so (39) is satisfied. In terms of $\xi_1^{[\alpha]}$ (52) can be written

$$\Pi_1[F_1, \zeta_1]|_x = \sum_{[\alpha]} q^{[\alpha]} \star \int_{\pi_X} \xi_1^{[\alpha]} + \star d(\Xi_1[F_1])|_x + \star d(\check{Z}_1[\zeta_1])$$

Then from (5)

$$\begin{aligned}
d \star \Pi_1[F_1, \zeta_1] &= d \star \left(\sum_{[\alpha]} q^{[\alpha]} \oint_{\pi_X} \xi_1^{[\alpha]} \right) \\
&= - \sum_{[\alpha]} q^{[\alpha]} d \oint_{\pi_X} \xi_1^{[\alpha]} = - \sum_{[\alpha]} q^{[\alpha]} \oint_{\pi_X} d\xi_1^{[\alpha]} \\
&= - \sum_{[\alpha]} q^{[\alpha]} \oint_{\pi_X} \theta_1^{[\alpha]}
\end{aligned}$$

Thus the Maxwell equation (44) is also satisfied. That (52) is the general solution to (44) follows from the fact that the difference between any two solutions of (44) satisfies the homogeneous differential equation associated with (44).

Thus we have succeeded in eliminating $\theta_1^{[\alpha]}$ from the perturbation system (38-41), thereby reducing the system to $dF_1 = 0$ and

$$\epsilon_0 d \star F_1 + \frac{1}{2} \sum_{[\alpha]} q^{[\alpha]} d \oint_{\pi_X} \oint_{\varpi_X^{[\alpha]}} d\tau \wedge \phi^{[\alpha]\star}(i_{\hat{W}_1^{[\alpha]}(F_1)} \theta_0^{[\alpha]}) + \sum_{[\alpha]} q^{[\alpha]} d \oint_{\pi_X} \varphi^{[\alpha]}(\zeta_1^{[\alpha]}) = 0 \quad (58)$$

in terms of (θ_0, F_0) , for the perturbation F_1 . The perturbation θ_1 is then given by (57,56).

C. The susceptibility kernel for an unbounded collisionless plasma

Equating (52) and (45) with the initial data

$$Z_1[\zeta_1] = \sum_{[\alpha]} q^{[\alpha]} \star \oint_{\pi_X} \varphi^{[\alpha]}(\zeta_1^{[\alpha]}) + \star d(\check{Z}_1[\zeta_1]) \quad (59)$$

yields

$$\oint_{p_X} \chi \wedge p_Y^\star(F_1) = \sum_{[\alpha]} q^{[\alpha]} \star \oint_{\pi_X} \oint_{\varpi_X^{[\alpha]}} d\tau \wedge \phi^{[\alpha]\star}(i_{\hat{W}_1^{[\alpha]}(F_1)} \theta_0^{[\alpha]}) + \star d(\Xi_1[F_1]) \quad (60)$$

Away from the initial hypersurface boundary $\partial(M_X^+ \times M_Y^+) = \Sigma_{M_X} \times M_Y^+ \cup M_X^+ \times \Sigma_{M_Y}$, using (5) and (A2) one has

$$\oint_{p_X} \star_X d_X \check{\xi} \wedge p_Y^\star(F_1) = \oint_{p_X} \star_X d\check{\xi} \wedge p_Y^\star(F_1) = \star d \oint_{p_X} \check{\xi} \wedge p_Y^\star(F_1) = \star d(\check{\Xi}_1[F_1])$$

where $\check{\Xi}_1[F_1]$ is a linear functional of F_1 . The gauge freedom $\chi \rightarrow \star_X d_X \check{\xi}$ given in (10) is equivalent to the addition of the term $\star d(\Xi_1[F_1])$ in (52).

If F_1 is restricted to have support in a certain domain one may find χ such that

$$\oint_{p_X} \chi \wedge p_Y^\star(F_1) = \sum_{[\alpha]} q^{[\alpha]} \star \oint_{\pi_X} \oint_{\varpi_X^{[\alpha]}} d\tau \wedge \phi^{[\alpha]\star}(i_{\hat{W}_1^{[\alpha]}(F_1)} \theta_0^{[\alpha]}) \quad (61)$$

To find such a susceptibility kernel requires the following maps.

For $(y, u) \in \mathcal{E}_Y^+$, let $C_{(y,u)}^{[\alpha]} : \mathbb{R}^+ \rightarrow M^+$ and $\dot{C}_{(y,u)}^{[\alpha]} : [0, \tau_1^{[\alpha]}(y, u)) \rightarrow \mathcal{E}^+$ be the unique solutions to the unperturbed Lorentz force equation (46,47) with initial conditions

$$C_{(y,u)}^{[\alpha]}(0) = y \quad \text{and} \quad \dot{C}_{(y,u)}^{[\alpha]}(0) = (y, u) \quad (62)$$

where $\tau_1^{[\alpha]}(y, u) \in \mathbb{R}^+ \cup \{\infty\}$ is the supremum of the values of τ such that $C_{(y,u)}^{[\alpha]}(\tau) \in M$. Let $\Phi^{[\alpha]} : \mathcal{N}_Y^{[\alpha]} \rightarrow M_X^+ \times M_Y^+$,

$$\Phi^{[\alpha]}(\tau, y, u) = (C_{(y,u)}^{[\alpha]}(\tau), y) \quad (63)$$

where

$$\mathcal{N}_Y^{[\alpha]} = \{(\tau, y, u) \in \mathbb{R}^+ \times \mathcal{E}_X^+ \mid 0 \leq \tau < \tau_1^{[\alpha]}(y, u)\}$$

This map gives the final and initial positions of a solution to the unperturbed Lorentz force equation in terms of the initial position, velocity and proper time parameter $\tau \in [0, \tau_1^{[\alpha]}(y, u))$.

Observe that $\Phi^{[\alpha]}$ is never surjective, since if $\Phi^{[\alpha]}(\tau, y, u) = (x, y)$ then $x \in J^+(y)$. Also $\Phi^{[\alpha]}$ is never injective since $\Phi^{[\alpha]}(0, y, u) = (y, y)$ for all $(y, u) \in \mathcal{E}_Y^+$. Thus $\Phi^{[\alpha]}$ does not possess an inverse and one must work locally on $M_X^+ \times M_Y^+$ in order to establish the diffeomorphism $\Psi^{[\alpha]} : \mathcal{D} \rightarrow \mathcal{D}'$,

$$\Psi^{[\alpha]} = (\Phi^{[\alpha]}|_{\mathcal{D}'})^{-1} \quad (64)$$

i.e.

$$\Psi^{[\alpha]}(C_{(y,u)}^{[\alpha]}(\tau), y) = (\tau, y, u)$$

with $\mathcal{D} \subset M_X^+ \times M_Y^+$ and $\mathcal{D}' \subset \mathcal{N}_Y^{[\alpha]}$ given by

$$\mathcal{D} = \left\{ (x, y) \left| \text{There exists a unique } u \in \mathcal{E}_y \text{ and } \tau \in \mathbb{R}^+ \text{ such that } C_{(y,u)}^{[\alpha]}(\tau) = x \text{ for all } [\alpha] \right. \right\} \quad (65)$$

and

$$\mathcal{D}' = \left\{ (\tau, y, u) \left| \Phi^{[\alpha]}(\tau, y, u) \in \mathcal{D} \text{ for all } [\alpha] \right. \right\}$$

This map $\Psi^{[\alpha]}$ encodes the solution to the two-point problem, namely given an initial event $y \in M_Y$ and final event $x \in M_X$ find the unique worldline to the unperturbed Lorentz force equation which passes through these two points. This worldline is specified by its initial velocity $(y, u) \in \mathcal{E}_X^+$ and its proper time τ . The statement that $\Phi^{[\alpha]}$ does not have an inverse is equivalent to the statement that in general there may not be a unique solution to the two point problem on an arbitrary domain. The domain \mathcal{D} is the set of all pairs (x, y) such that there is a unique worldline.

Set

$$\chi = \sum_{[\alpha]} \chi^{[\alpha]} \quad (66)$$

where

$$\chi^{[\alpha]}|_{(x,y)} = \frac{1}{2} \frac{q^{[\alpha]2}}{m^{[\alpha]}} \star_X dy^{cd} \wedge i_{abcd}^{(y)} \Psi^{[\alpha]\star} \left(d\tau \wedge \varpi_Y^{[\alpha]\star} (g^{\nu a} u^b i_{\nu}^{(u)} \theta_0^{[\alpha]}) \right) \Big|_{(x,y)} \quad (67)$$

for points $(x, y) \in \mathcal{D}$. In the appendix (lemma 6) it is shown that given $x \in M_X^+$ (61) and F_1 with support in

$$\mathcal{D}_x = \mathcal{D} \cap p_X^{-1}\{x\} = \{y \in M_Y | (x, y) \in \mathcal{D}\} \quad (68)$$

then (61) holds at x . Furthermore although $(d_Y \chi)|_{(x,y)}$ is unique, χ has the gauge freedom given by (9).

One may write (67) implicitly as

$$\chi^{[\alpha]} \wedge p_Y^{\star} \gamma = -q^{[\alpha]} \star_X S \Psi^{[\alpha]\star} \left(d\tau \wedge \varpi_Y^{[\alpha]\star} (i_{\hat{W}_1^{[\alpha]}(\gamma)} \theta_0^{[\alpha]}) \right) \quad (69)$$

for all $\gamma \in \Gamma \Lambda^2 M_Y^+$ where $S : \Lambda_{(x,y)}^6(M_X^+ \times M_Y^+) \rightarrow \Lambda_{(x,y)}^6(M_X^+ \times M_Y^+)$,

$$S(\alpha) = i_{0123}^{(y)} \alpha \wedge dy^{0123} \quad (70)$$

The tensor projector S has the simplest representation in the coordinate basis employed here since $i_a^{(y)} dy^b = \delta_a^b$.

From (64) for a chosen species $[\alpha]$ one must consider τ and u to be functions of (x, y) as well as the species label $[\alpha]$. Thus let $\Psi^{[\alpha]}$ be given by the functions $\tau = \tau(x, y)$ and $u^\mu = u^\mu(x, y)$, where we have dropped the species label, i.e. $\tau(x, y)$ and $u^\mu(x, y)$ solve the implicit equation

$$C_{(y, u(x, y))}^{[\alpha]}(\tau(x, y)) = x \quad (71)$$

where $u^0(x, y)$ is the solution to $u^a(x, y)u^b(x, y)g_{ab}(y) = -1$ and $u_0(x, y) = g_{a0}(y)u^a(x, y)$. Let $f_0^{[\alpha]} = f_0^{[\alpha]}(y, u)$ represent the unperturbed probability function on \mathcal{E}_Y^+ . The contribution to the susceptibility kernel from species $[\alpha]$ is given in local coordinates by (lemma 7 in appendix.)

$$\begin{aligned} \chi^{[\alpha]}|_{(x,y)} = & -f_0^{[\alpha]} \frac{q^{[\alpha]2}}{m^{[\alpha]}} \frac{|\det g|^{3/2}}{4u_0} g^{\mu c} u^b \epsilon^{dejk} \epsilon_{cbih} \epsilon_{\mu\nu\sigma} \times \\ & \left(\frac{u^a}{2} \frac{\partial \tau}{\partial y^a} \frac{\partial u^\nu}{\partial x^d} \frac{\partial u^\sigma}{\partial x^e} - \frac{u^a}{2} \frac{\partial \tau}{\partial x^d} \frac{\partial u^\nu}{\partial y^a} \frac{\partial u^\sigma}{\partial x^e} + \frac{u^a}{2} \frac{\partial \tau}{\partial x^d} \frac{\partial u^\nu}{\partial x^e} \frac{\partial u^\sigma}{\partial y^a} \right. \\ & \left. + \left(-\Gamma_{pf}^\nu u^p u^f + \frac{q^{[\alpha]}}{m^{[\alpha]}} F_{0pf} g^{\nu p} u^f \right) \frac{\partial \tau}{\partial x^d} \frac{\partial u^\sigma}{\partial x^e} \right) dx_{jk} \wedge dy^{ih} \end{aligned} \quad (72)$$

where g , F_0 and Γ_{ef}^ν are all evaluated at $y \in M_Y^+$ and each τ and u belongs to the species $[\alpha]$. This is a key result of our article.

D. A spacetime inhomogeneous microscopically neutral plasma.

In a Vlasov model, a plasma or gas is deemed *microscopically neutral* if in its unperturbed state $F_0 = 0$. Let M be Minkowski spacetime with global Lorentzian coordinates so that $\Gamma_{ab}^\nu = 0$. Assume that $f_0^{[\alpha]}$ solves the zeroth order Maxwell-Vlasov system (35) with $\theta_0^{[\alpha]} = i_{W_0^{[\alpha]}}(f_0^{[\alpha]}\Omega)$ and $F_0 = 0$. In this scenario one can calculate χ explicitly.

Since Minkowski spacetime is flat and $F_0 = 0$ the integral curves $C_{(x,v)}$ in global Lorentzian coordinates are the straight lines:

$$\tau = \sqrt{-g(x-y, x-y)} \quad \text{and} \quad u = \frac{(x-y)}{\tau} \quad (73)$$

Differentiating with respect to x^a and y^a gives.

$$\begin{aligned} \frac{\partial \tau}{\partial x^a} &= -u_a, & \frac{\partial \tau}{\partial y^a} &= u_a, & \frac{\partial u^a}{\partial x^b} &= \frac{(\delta_b^a + u_a u_b)}{\tau} \\ \text{and} & & \frac{\partial u^a}{\partial y^b} &= -\frac{(\delta_b^a + u_a u_b)}{\tau} \end{aligned} \quad (74)$$

It follows from (72) that

$$\chi^{[\alpha]}|_{(x,y)} = \frac{q^{[\alpha]} f_0^{[\alpha]}(y, u)}{4u_0 \tau^2} g^{\mu c} u^b \epsilon_{cbih} (2dx_{0\mu} + \epsilon^{d\sigma jk} \epsilon_{\mu\nu\sigma} u^\nu u_d dx_{jk}) \wedge dy^{ih} \quad (75)$$

where $\tau(x, y)$ and $u(x, y)$ are given by (73).

It is often useful to explore the response of an inhomogeneous plasma due to a monochromatic electromagnetic plane wave with constant amplitude E :

$$F_1 = E e^{-i\omega x^0 + i k x^1} dx^{01}. \quad (76)$$

Setting the initial hypersurface as $\Sigma_{\mathcal{E}_Y} = \{y^0 = y_0^0\}$, the general initial 5-form $\zeta_1^{[\alpha]} \in \Gamma \Lambda_{\Sigma_{\mathcal{E}_Y}}^5 \mathcal{E}_Y^+$ satisfying $i_{W_0} \zeta_1^{[\alpha]} = 0$ is given in terms of its components by

$$\begin{aligned} \zeta_1^{[\alpha]}|_{(0,y^\mu,u^\nu)} &= (u^0 dy^1 - u^1 dy^0) \wedge (\zeta_{1,1}^{[\alpha]} dy^2 \wedge du^{123} + \zeta_{1,2}^{[\alpha]} dy^3 \wedge du^{123}) + \zeta_{1,3}^{[\alpha]} dy^{23} \wedge du^{123} \\ &\quad + (u^0 dy^{123} - u^1 dy^{023}) (\zeta_{1,4}^{[\alpha]} du^{12} + \zeta_{1,5}^{[\alpha]} du^{13} + \zeta_{1,6}^{[\alpha]} du^{23}) \end{aligned} \quad (77)$$

where $\zeta_{1,A}^{[\alpha]} = \check{\zeta}_{1,A}^{[\alpha]}(y^\mu, u^\nu)$ for $A = 1, \dots, 6$. For the integral curves (73) and the initial hypersurface $\Sigma_{\mathcal{E}_Y} = \{y^0 = y_0^0\}$ one has $\tau_0(x, v) = (y_0^0 - x^0)/v^0$ and the map φ is given by (51) with $\phi_\tau^*(y^a) = x^a + \tau y^a$ and $\phi_\tau^*(u^a) = v^a$. From (45) with χ given by (75) and $Z_1[\zeta_1]$ given by (59) one has:

$$\begin{aligned} \Pi_1[F_1, \zeta_1] &= \\ &- \sum_{[\alpha]} \frac{q^{[\alpha]^2}}{m^{[\alpha]}} E e^{-i\omega x^0 + i k x^1} \left\{ dx^{01} \int dv^{123} T^{[\alpha]} \frac{(v^0)^2 - (v^1)^2}{v^0} + dx^{12} \int dv^{123} T^{[\alpha]} v^2 \right. \\ &\quad \left. - dx^{02} \int dv^{123} T^{[\alpha]} \frac{v^2 v^1}{v^0} + dx^{13} \int dv^{123} T^{[\alpha]} v^3 + dx^{03} \int dv^{123} T^{[\alpha]} \frac{v^3 v^1}{v^0} \right\} \\ &+ \sum_{[\alpha]} q^{[\alpha]} \left\{ dx^{02} \int dv^{123} \left(\zeta_{1,4}^{[\alpha]} \frac{v^1(x^0 - y_0^0)}{v^0} - \zeta_{1,1}^{[\alpha]} v^1 \right) + dx^{03} \int dv^{123} \left(\zeta_{1,5}^{[\alpha]} \frac{v^1(x^0 - y_0^0)}{v^0} - \zeta_{1,2}^{[\alpha]} v^1 \right) \right. \\ &\quad + dx^{12} \int dv^{123} \left(v^0 \zeta_{1,1}^{[\alpha]} - \zeta_{1,4}^{[\alpha]}(x^0 - y_0^0) \right) + dx^{13} \int dv^{123} \left(v^0 \zeta_{1,2}^{[\alpha]} - \zeta_{1,5}^{[\alpha]}(x^0 - y_0^0) \right) \\ &\quad \left. + dx^{23} \int dv^{123} \left(\zeta_{1,3}^{[\alpha]} + \zeta_{1,4}^{[\alpha]} \frac{v^1 v^3 (x^0 - y_0^0)}{(v^0)^2} - \zeta_{1,5}^{[\alpha]} \frac{v^1 v^2 (x^0 - y_0^0)}{(v^0)^2} + \zeta_{1,6}^{[\alpha]}(x^0 - y_0^0) \left(\frac{v^1}{v^0} - 1 \right) \right) \right\} \\ &+ \star d(\Xi_1[F_1]) + \star d(\check{Z}_1[\zeta_1]) \end{aligned} \quad (78)$$

where $\int dv^{123}$ denotes the triple integral operator $\iiint_{-\infty}^{\infty} dv^{123}$, $v^0 = \sqrt{1 + v_\mu v^\mu}$,

$$T^{[\alpha]} = T^{[\alpha]}(x, v) = \int_{(y_0^0 - x^0)/v^0}^0 e^{i\tau(-\omega v^0 + k v^1)} f_0^{[\alpha]}(x + \tau v, v) \tau d\tau \quad (79)$$

and $\zeta_{1,A}^{[\alpha]} = \check{\zeta}_{1,A}^{[\alpha]}(x^\mu, v^\mu) = \check{\zeta}_{1,A}^{[\alpha]}(x^\mu - x^0 v^\mu / v^0, v^\mu)$ in (78). This response is not in general plane fronted.

For the particular case of a plane fronted plasma distribution:

$$f_0^{[\alpha]}(x, v) = h_0^{[\alpha]}(x^0, x^1, v^1) \delta(v^2) \delta(v^3) \quad (80)$$

with initial data:

$$\zeta_1^{[\alpha]} = 0$$

(78) becomes the plane fronted 2-form

$$\begin{aligned} & \Pi_1[F_1, \zeta] \big|_x \\ &= -dx^{01} \sum_{[\alpha]} \frac{q^{[\alpha]2}}{m^{[\alpha]}} E e^{-i\omega x^0 + ikx^1} \int_{-\infty}^{\infty} dv^1 \int_{(y_0^0 - x^0)/v^0}^0 d\tau e^{i\tau(-\omega v^0 + kv^1)} h_0^{[\alpha]}(x^0 + \tau v^0, x^1 + \tau v^1, v^1) \frac{\tau}{v^0} \\ & \quad + \star d(\Xi_1[F_1]) \end{aligned} \tag{81}$$

describing the response of a spacetime inhomogeneous unbounded plasma to (76).

E. Spacetime homogeneous unbounded plasmas

The previous discussion simplifies considerably if the unperturbed plasmas is homogeneous in space and time. In Minkowski spacetime M , an unbounded unperturbed plasma is deemed *spacetime homogeneous* if $A_z^\star F_0 = F_0$ and $\dot{A}_z^\star \theta_0^{[\alpha]} = \theta_0^{[\alpha]}$ for all $z \in M$ where the translation map $A_z : M \rightarrow M$, $A_z(x) = x + z$ induces the map $\dot{A}_z : \mathcal{E} \rightarrow \mathcal{E}$, $\dot{A}_z = A_{z\star}$. Such spacetime homogeneity implies that in all inertial frames the medium is stationary and spatially homogeneous in all directions. Such a spacetime homogeneous plasma will give rise to a spacetime homogeneous electromagnetic constitutive relation. In addition to the components $(F_0)_{ab}$ with respect to an inertial frame being constant, the functions $f^{[\alpha]}(x, v)$ are independent of event position x and can therefore be written $f^{[\alpha]}(v)$.

In this scenario the Fourier transform (17) of the susceptibility kernel (18) for each species, is then given by

$$\begin{aligned} & \hat{\chi}^{[\alpha]}_{ab}{}^{ef}(k) dx^{ab} \\ &= \frac{1}{2} q^{[\alpha]} dx_{gh} \int_{-\infty}^0 d\tau \int dv^{123} f_0^{[\alpha]}(v) e^{-ik \cdot \mathbf{L}^{[\alpha]} v} \frac{v^g}{v_0} (g^{\nu e} u^f - g^{\nu f} u^e) \left(\mathbf{L}^{[\alpha]}{}_{\nu}{}^h(\tau) - \frac{u_\nu}{u_0} \mathbf{L}^{[\alpha]}{}_0{}^h(\tau) \right) \end{aligned} \tag{82}$$

where \mathbf{F}_0 is the 4×4 real matrix with components $(\mathbf{F}_0)^a{}_b = \eta^{ac}(F_0)_{cb}$ generating the matrices

$$\begin{aligned} \mathbf{D}^{[\alpha]}{}^a{}_b(\tau) &= \exp \left(\tau \frac{q^{[\alpha]}}{m^{[\alpha]}} \mathbf{F}_0 \right)^a{}_b, & \mathbf{D}^{[\alpha]}{}_b{}^a(\tau) &= g_{bc} \mathbf{D}^{[\alpha]}{}^c{}_d(\tau) g^{da}, \\ \mathbf{L}^{[\alpha]}{}^a{}_b(\tau) &= \int_0^\tau \mathbf{D}^{[\alpha]}{}^a{}_b(\tau') d\tau', & \mathbf{L}^{[\alpha]}{}_b{}^a(\tau) &= g_{bc} \mathbf{L}^{[\alpha]}{}^c{}_d(\tau) g^{da}, \end{aligned} \tag{83}$$

$$k \cdot \mathbf{L}^{[\alpha]} v = k_a \mathbf{L}^{[\alpha]a}{}_b(\tau) v^b \text{ and}$$

$$u^a(\tau, v^1, v^2, v^3) = \mathbf{D}^{[\alpha]a}{}_b(\tau) v^b \quad (84)$$

The susceptibility kernel (82) can be shown to agree with the results of O’Sullivan and Derfler¹².

Furthermore for a microscopically neutral spacetime homogeneous plasma with $F_0 = 0$, $G_1 = 0$ and $f_0^{[\alpha]}(v) = h_0^{[\alpha]}(v^1)\delta(v^2)\delta(v^3)$ it follows from (81) and (43) that for $\text{Im}(\omega) > 0$

$$1 = \sum_{[\alpha]} \frac{q^{[\alpha]2}}{m^{[\alpha]}\epsilon_0} \int_{-\infty}^{\infty} \frac{h_0^{[\alpha]}(v^1) dv^1}{v^0(-\omega v^0 + kv^1)^2} \quad (85)$$

The relativistic Landau damped dispersion relation for plane fronted Langmuir modes in an unperturbed spacetime homogeneous plasma arises by analytic continuation of the integral (85) to the lower-half complex ω plane.

F. Langmuir modes for an inhomogeneous unbounded plasma in Minkowski spacetime

If the plasma is microscopically neutral but spacetime inhomogeneous in its unperturbed state the Landau dispersion relation corresponding to (85) becomes more involved. We define the generalized Langmuir sector to contain perturbations described by (81) but with the external polarization specified by $\Xi_1[F_1]$ set to zero. Since $\zeta_1^{[\alpha]} = 0$, $\Pi_1[F_1, 0]$ will be denoted $\Pi_1[F_1]$. Thus (43) with $G_1 = 0$ becomes

$$\epsilon_0 F_1 = -\Pi_1[F_1] \quad (86)$$

Consider the case where planar inhomogeneities in a plasma composed of electrons and ions arise from the unperturbed spacetime inhomogeneous solution to the Maxwell-Vlasov system: (35-36) with $F_0 = 0$ and

$$\begin{aligned} f_0^{[\text{el}]}(x^0, x^1, x^2, x^3, v^1, v^2, v^3) &= f_0^{[\text{ion}]}(x^0, x^1, x^2, x^3, v^1, v^2, v^3) \\ &= h\left(x^1 - \frac{v^1 x^0}{v^0}, v^1\right) \delta(v^2) \delta(v^3) \end{aligned} \quad (87)$$

where $q^{[\text{el}]} = -q^{[\text{ion}]}$.

For example one might consider

$$h(x^1, v^1) = n^{[\text{ion}]}(x^1) A^{[\text{ion}]}(x^1) \exp\left(-\frac{m^{[\text{ion}]} v^0}{k_B T^{[\text{ion}]}(x^1)}\right)$$

where $A^{[\text{ion}]}(x^1)$ normalizes (87). Then $f^{[\text{ion}]}$ initially at $x^0 = 0$ represents a distribution of ions where, at each spatial point x^1 , the velocities belong to the 1-dimensional Maxwell-Jüttner distribution. In such a distribution the temperature $T^{[\text{ion}]}(x^1)$ and the number density of ions $n^{[\text{ion}]}(x^1)$ depend on position. It follows from (87) that $f^{[\text{el}]}$ also initially represents a position dependent Maxwell-Jüttner distribution where $n^{[\text{el}]}(x^1) = n^{[\text{ion}]}(x^1)$ and $T^{[\text{el}]}(x^1) = T^{[\text{ion}]}(x^1)m^{[\text{el}]} / m^{[\text{ion}]}$. After the initial moment, the ions and electrons drift according to (87) and velocities do not remain in the Maxwell-Jüttner distributions. Alternatively (87) might describe a plasma composed of particles and anti-particles.

In the theory of a spacetime homogeneous plasma ω and k satisfy the transcendental dispersion relation (85). This relation contains an integral that is potentially singular. The Landau prescription circumvents this singularity by complexifying ω and defining an analytic continuation for the integral in the complex ω plane.

Setting $h_0^{[\text{el}]}(x^0, x^1, v^1) = h(x^1 - v^1 x^0 / v^0, v^1)$ in (80) yields (87) and (81) becomes

$$\begin{aligned} \Pi_1[F_1]|_x &= -dx^{01} q^{[\text{el}]^2} \left(\frac{1}{m^{[\text{ion}]}} + \frac{1}{m^{[\text{el}]}} \right) E e^{-i\omega x^0 + ikx^1} \int_{-\infty}^{\infty} dv^1 h\left(x^1 - \frac{v^1 x^0}{v^0}, v^1\right) \int_{(y_0^0 - x^0)/v^0}^0 d\tau e^{i\tau(-\omega v^0 + kv^1)} \frac{\tau}{v^0} \end{aligned} \quad (88)$$

To compare with the results (85) given for the homogeneous case, consider the limit $y_0^0 \rightarrow -\infty$ with $\text{Im}(\omega) > 0$. Furthermore for the non-evanescent modes considered here $\text{Im}(k) = 0$. Thus (88) becomes

$$\Pi_1[F_1]|_x = -dx^{01} \epsilon_0 \mathcal{Q}_0^2 E e^{-i\omega x^0 + ikx^1} \int_{-\infty}^{\infty} dv^1 \frac{h(x^1 - v^1 x^0 / v^0, v^1)}{v^0 (-\omega v^0 + kv^1)^2} \quad (89)$$

where

$$\mathcal{Q}_0^2 = \frac{q^{[\text{el}]^2}}{\epsilon_0 m^{[\text{ion}]}} + \frac{q^{[\text{el}]^2}}{\epsilon_0 m^{[\text{el}]}}$$

In a spacetime inhomogeneous plasma there is no time-harmonic solution or associated transcendental dispersion relation between ω and k . We therefore propose solving (86) with a longitudinal field F_1 represented as the packet

$$F_1(x^0, x^1) = dx^{01} \int_{-\infty}^{\infty} d\hat{\omega} \int_{-\infty}^{\infty} d\hat{k} \hat{E}(\hat{\omega}, \hat{k}) e^{-i\hat{\omega}x^0 + i\hat{k}x^1} \quad (90)$$

Substituting (89) and (90) into (86) yields

$$\begin{aligned} & \int_{-\infty}^{\infty} d\hat{\omega} \int_{-\infty}^{\infty} d\hat{k} \hat{E}(\hat{\omega}, \hat{k}) e^{-i\hat{\omega}x^0 + i\hat{k}x^1} \\ &= \mathcal{Q}_0^2 \int_{-\infty}^{\infty} d\hat{\omega} \int_{-\infty}^{\infty} d\hat{k} \hat{E}(\hat{\omega}, \hat{k}) e^{-i\hat{\omega}x^0 + i\hat{k}x^1} \int_{-\infty}^{\infty} dv^1 \frac{h(x^1 - v^1 x^0/v^0, v^1)}{v^0(\hat{\omega}v^0 + \hat{k}v^1)^2} \end{aligned}$$

Performing the inverse Fourier transform gives

$$\begin{aligned} & 4\pi^2 \hat{E}(\omega, k) \\ &= \mathcal{Q}_0^2 \int_{-\infty}^{\infty} dx^0 \int_{-\infty}^{\infty} dx^1 \int_{-\infty}^{\infty} d\hat{\omega} \int_{-\infty}^{\infty} d\hat{k} \hat{E}(\hat{\omega}, \hat{k}) e^{i(-(\hat{\omega}-\omega)x^0 + (\hat{k}-k)x^1)} \int_{-\infty}^{\infty} dv^1 \frac{h(x^1 - v^1 x^0/v^0, v^1)}{v^0(\hat{\omega}v^0 + \hat{k}v^1)^2} \end{aligned}$$

Since

$$\int_{-\infty}^{\infty} dx^0 \int_{-\infty}^{\infty} dx^1 e^{i(-(\hat{\omega}-\omega)x^0 + (\hat{k}-k)x^1)} h(x^1 - v^1 x^0/v^0, v^1) = 2\pi \hat{h}(k - \hat{k}, v^1) \delta(\hat{\omega} - \omega + v^1(k - \hat{k})/v^0)$$

where

$$\hat{h}(k, v^1) = \int_{-\infty}^{\infty} e^{-iks} h(s, v^1) ds$$

one has

$$\hat{E}(\omega, k) = \frac{\mathcal{Q}_0^2}{2\pi} \int_{-\infty}^{\infty} d\hat{\omega} \int_{-\infty}^{\infty} d\hat{k} \int_{-\infty}^{\infty} dv^1 \frac{\hat{E}(\hat{\omega}, \hat{k})}{v^0(\hat{\omega}v^0 + \hat{k}v^1)^2} \hat{h}(k - \hat{k}, v^1) \delta(\hat{\omega} - \omega + v^1(k - \hat{k})/v^0) \quad (91)$$

Since we restrict to non-evanescent modes k and \hat{k} are real. For $\hat{E}(\omega, k)$ to be non-zero one requires the argument of the δ -function to be zero. Since v^1 is real and therefore $v^1(k - \hat{k})/v^0$ is real it follows that although $\text{Im}(\omega) > 0$ and $\text{Im}(\hat{\omega}) > 0$ the difference $\omega - \hat{\omega}$ is real. Furthermore from $\hat{\omega} - \omega + v^1(k - \hat{k})/v^0 = 0$ it follows that $|\hat{\omega} - \omega| < |\hat{k} - k|$. Thus (91) becomes

$$\hat{E}(\omega, k) = \frac{\mathcal{Q}_0^2}{2\pi} \int_{-\infty}^{\infty} d\hat{k} I(\omega, k, \hat{k}) \quad (92)$$

where

$$I(\omega, k, \hat{k}) = \int_{S(\omega, k, \hat{k})} d\hat{\omega} \hat{E}(\hat{\omega}, \hat{k}) \frac{(k - \hat{k})}{(\hat{\omega}k - \hat{k}\omega)^2} \hat{h}\left(k - \hat{k}, \frac{k - \hat{k}}{\sqrt{(\hat{k} - k)^2 - (\hat{\omega} - \omega)^2}}\right) \quad (93)$$

and the contour of integration for $\hat{\omega}$ in (93) is the straight line $S(\omega, k, \hat{k})$ where $\text{Im}(\hat{\omega}) = \text{Im}(\omega) > 0$ and $-|\hat{k} - k| < \text{Re}(\hat{\omega} - \omega) < |\hat{k} - k|$. Since $(\hat{\omega} - \omega)^2 < (\hat{k} - k)^2$ the arguments of \hat{h} in (93) are always real and non-singular on $S(\omega, k, \hat{k})$.

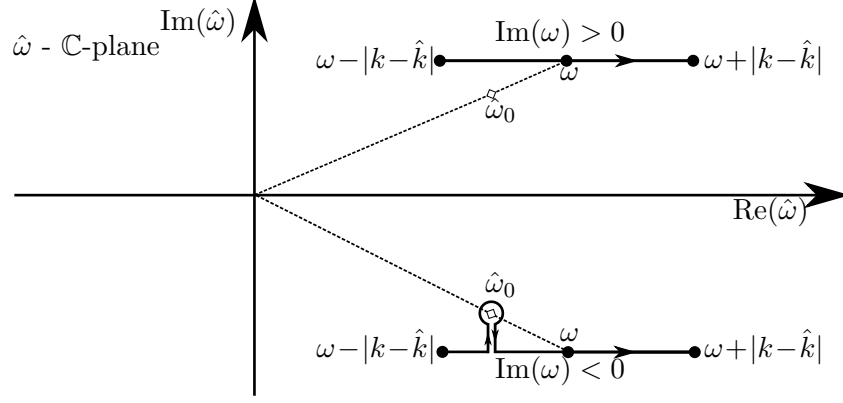


FIG. 2. The upper contour denotes $S(\omega, k, \hat{k})$ when $\text{Im}(\omega) > 0$ for real k, \hat{k} . The lower contour of integration is used when $\text{Im}(\omega) < 0$ for real k, \hat{k} .

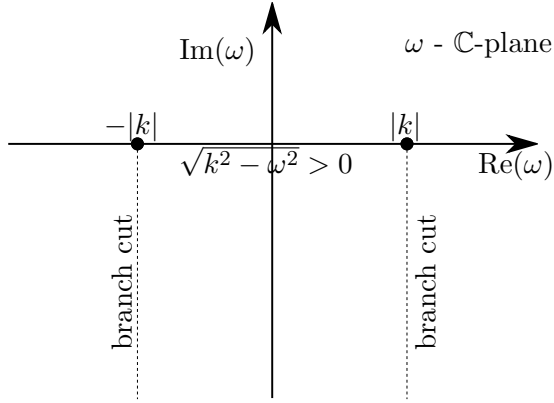


FIG. 3. Branch cuts in ω for $I(\omega, k, \hat{k})$.

To accommodate the situation when $\hat{E}(\omega, k)$ describes damped electromagnetic waves one must continue (93) to $\text{Im}(\omega) < 0$ for real k . However there is a double pole in the complex $\hat{\omega}$ plane at $\hat{\omega} = \hat{\omega}_0 = \hat{k}\omega/k$ that coincides with $S(\omega, k, \hat{k})$ when $\text{Im}(\omega) = 0$ and $|\omega| < |k|$. To define an analytic continuation of (93) to $\text{Im}(\omega) < 0$ when $|\text{Re}(\omega)| < |k|$, we indent $S(\omega, k, \hat{k})$ to encircle the pole in the standard manner and write the contour integral in terms of a principle part and associated residue, see figure 2. Such a continuation scheme gives rise to branches in the ω plane for $I(\omega, k, \hat{k})$ as shown in figure 3.

This analytic continuation of (93) to $\text{Im}(\omega) < 0$ acquires the residue

$$R(\omega, k, \hat{k}) = \frac{|k - \hat{k}|}{k|k|} \frac{\partial \hat{E}}{\partial \omega} \left(\frac{\omega \hat{k}}{k}, \hat{k} \right) \hat{h} \left(k - \hat{k}, \frac{s_k s_{k-\hat{k}} \omega}{\sqrt{k^2 - \omega^2}} \right) \\ - \frac{k}{(k^2 - \omega^2)^{3/2}} \hat{E} \left(\frac{\omega \hat{k}}{k}, \hat{k} \right) \hat{h}_{v^1} \left(k - \hat{k}, \frac{s_k s_{k-\hat{k}} \omega}{\sqrt{k^2 - \omega^2}} \right)$$

where $\hat{h}_{v^1}(k, v^1) = \frac{\partial \hat{h}}{\partial v^1}(k, v^1)$, $s_k = k/|k|$ and $s_{k-\hat{k}} = (k - \hat{k})/|k - \hat{k}|$. In the case when $\text{Im}(\omega) = 0$, the principle value of (93) is taken together with residue $\frac{1}{2}R(\omega, k, \hat{k})$. Equation (92) then gives

$$\hat{E}(\omega, k) = \frac{k}{2\pi} \int_{-\infty}^{\infty} I(\omega, k, \hat{k}) d\hat{k} \quad \text{if } \text{Im}(\omega) > 0 \quad \text{or} \quad |\text{Re}(\omega)| > |k| \\ \hat{E}(\omega, k) = \frac{k}{2\pi} \int_{-\infty}^{\infty} I(\omega, k, \hat{k}) d\hat{k} - ik \int_{-\infty}^{\infty} R(\omega, k, \hat{k}) d\hat{k} \\ \text{if } \text{Im}(\omega) < 0 \quad \text{and} \quad |\text{Re}(\omega)| \leq |k| \\ \hat{E}(\omega, k) = \frac{k}{2\pi} \int_{-\infty}^{\infty} \mathcal{P}I(\omega, k, \hat{k}) d\hat{k} - \frac{ik}{2} \int_{-\infty}^{\infty} R(\omega, k, \hat{k}) d\hat{k} \\ \text{if } \text{Im}(\omega) = 0 \quad \text{and} \quad |\text{Re}(\omega)| < |k| \quad (94)$$

where $\mathcal{P}I(\omega, k, \hat{k})$ in (94) refers to the principle part of (93) when $\text{Im}(\omega) = 0$ and $|\text{Re}(\omega)| < |k|$ and hence the pole at $\hat{\omega}_0$ lies on the contour $S(\omega, k, \hat{k})$. Thus in each domain above, the perturbation $\hat{E}(\omega, k)$ must be determined by solving a non-standard integral equation.

IV. CONCLUSIONS

In this article a classical covariant description of electromagnetic interactions in continuous matter in an arbitrary background gravitational field has been formulated in terms of a polarization 2-form that enters into the macroscopic Maxwell equations. Linear dispersive constitutive relations arise when this 2-form is expressed as an affine functional of the Maxwell 2-form with the aid of a 2-point susceptibility kernel. We have explored the constraints on this kernel imposed by causality requirements, spacetime Killing symmetries and local gauge freedoms. The formalism has been applied to an analysis of constitutive models for waves in collisionless plasmas. In particular a formula for the linear susceptibility of a fully ionized inhomogeneous unbounded non-stationary collisionless plasma to a perturbation in the presence of gravity has been given in terms of maps describing the dynamics

of the plasma. This formula has been elucidated by reference to both homogeneous and inhomogeneous perturbations in Minkowski spacetime. In the former case one recovers the standard Landau dispersion relation when perturbing Langmuir modes. In the latter case we have described a generalized damping mechanism for such modes that may arise when the unperturbed state is both inhomogeneous and non-stationary. Such a mechanism arises from the analytic continuation of an integral equation that replaces the Landau dispersion relation.

It is concluded that the use of a covariant 2-point affine susceptibility kernel in describing the electromagnetic response of dispersive media offers a modelling tool that naturally generalizes the use of permittivity and permeability tensors used to model electromagnetic interactions in non-relativistic media. The formulation in terms of an arbitrary background spacetime metric offers potential applications in a number of astrophysical contexts involving electromagnetic fields in inhomogeneous or non-stationary plasmas

ACKNOWLEDGEMENTS

The authors are grateful to support from EPSRC (EP/E001831/1) and the Cockcroft Institute (STFC ST/G008248/1).

REFERENCES

- ¹J. Gratus and R.W. Tucker. Covariant constitutive relations, Landau damping and non-stationary inhomogeneous plasmas. *Progress In Electromagnetics Research M*, 13:145–156, 2010.
- ²Points here refer in general to events in a spacetime manifold.
- ³In this article the term polarization will refer to any state of the medium that gives rise to magnetization or electrical polarization in some frame.
- ⁴R. Bott and L.W. Tu. *Differential forms in algebraic topology*. Springer, 1982.
- ⁵G. De Rham. *Differentiable manifolds: forms, currents, harmonic forms*. Springer Verlag, 1984.
- ⁶When A is not compact on M_Y invariance is modulo a boundary term.
- ⁷G.M. Wald. *General relativity*. Chicago, 1984.

⁸We write $y \in J^-(x)$ if x is (timelike or lightlike) causally connected to y and x lies in the future of y .

⁹Note that this definition of homogeneity refers only to the electromagnetic properties of a medium.

¹⁰K. Yano and S. Ishihara. *Tangent and cotangent bundles: differential geometry*. Dekker, 1973.

¹¹J. Ehlers. General relativity and kinetic theory. In R.K. Sachs, editor, *Course XLVII: General Relativity and Cosmology*, Proceedings of the International School of Physics E. Fermi:, pages 1–70, 1971.

¹²R.A. O’Sullivan and H. Derfler. Relativistic theory of electromagnetic susceptibility and its application to plasmas. *Physical Review A*, 8(5):2645–2656, 1973.

Appendix A: Proofs of results used used in the text.

Lemma 2. *Local representation of $\oint_{\pi_{\mathcal{N}}} \alpha$ in (4) from the implicit definition in equation (3).*

Proof. On a fibred manifold \mathcal{N} of dimension $n + r$ with projection $\pi_{\mathcal{N}} : \mathcal{N} \rightarrow N$ over a manifold N of dimension n . Thus at each point $\sigma \in N$ one has the fibre $\mathcal{N}_{\sigma} = \pi_{\mathcal{N}}^{-1}\{\sigma\} = \{(\sigma', \varsigma) \in \mathcal{N} \mid \pi_{\mathcal{N}}(\sigma', \varsigma) = \sigma\}$ so $\dim(\mathcal{N}_{\sigma}) = r$ is the fibre dimension. Let $(\sigma^1, \dots, \sigma^n)$ and $(\sigma^1, \dots, \sigma^n, \varsigma^1 \dots \varsigma^r)$ be local coordinates for patches on N and \mathcal{N} respectively.

Consider first the case when $\alpha \in \Gamma \Lambda^{p+r} \mathcal{N}$ consists of a single component $\alpha_I(\sigma, \varsigma) d\sigma^I \wedge d\varsigma^{1\dots r}$ with no sum on I . Hence explicit summation will be used in this particular proof. Set $\hat{I} = \{1, \dots, n\} \setminus I$ so that $d\sigma^{\hat{I}} \wedge d\sigma^I = \pm d\sigma^{1\dots n}$ and let $\beta = \sum_J \beta_J(\sigma) d\sigma^J$ then $\beta \wedge d\sigma^I =$

$\pm \beta_I \alpha_I d\sigma^{1\dots n}$ so that:

$$\begin{aligned}
& \sum_J \int_{(\sigma, \varsigma) \in \mathcal{N}} \pi_{\mathcal{N}}^*(\beta_J(\sigma) d\sigma^J) \wedge \alpha_I(\sigma, \varsigma) d\sigma^I \wedge d\varsigma^{1\dots r} \\
&= \sum_J \int_{(\sigma, \varsigma) \in \mathcal{N}} \beta_J(\sigma) d\sigma^J \wedge \alpha_I(\sigma, \varsigma) d\sigma^I \wedge d\varsigma^{1\dots r} \\
&= \sum_J \int_{(\sigma, \varsigma) \in \mathcal{N}} \beta_J(\sigma) d\sigma^J \wedge d\sigma^I \wedge \alpha_I(\sigma, \varsigma) d\varsigma^{1\dots r} \\
&= \int_{(\sigma, \varsigma) \in \mathcal{N}} \beta_{\hat{I}}(\sigma) d\sigma^{\hat{I}} \wedge d\sigma^I \wedge \alpha_I(\sigma, \varsigma) d\varsigma^{1\dots r} \\
&= \int_{\sigma \in N} \beta_{\hat{I}}(\sigma) d\sigma^{\hat{I}} \wedge d\sigma^I \int_{\mathcal{N}_\sigma} \alpha_I(\sigma, \varsigma) d\varsigma^{1\dots r} \\
&= \sum_J \int_{\sigma \in N} \beta_J(\sigma) d\sigma^J \wedge d\sigma^I \int_{\mathcal{N}_\sigma} \alpha_I(\sigma, \varsigma) d\varsigma^{1\dots r} \\
&= \int_{\sigma \in N} \beta \wedge d\sigma^I \int_{\mathcal{N}_\sigma} \alpha_I(\sigma, \varsigma) d\varsigma^{1\dots r}
\end{aligned}$$

Thus by linearity

$$\int_{\mathcal{N}} \pi_{\mathcal{N}}^*(\beta) \wedge \alpha = \sum_I \int_{\sigma \in N} \beta \wedge d\sigma^I \int_{\mathcal{N}_\sigma} \alpha_I(\sigma, \varsigma) d\varsigma^{1\dots r} \quad (\text{A1})$$

where $\alpha = \sum_I \alpha_I(\sigma, \varsigma) d\sigma^I \wedge d\varsigma^{1\dots r}$. If (4) holds then for $\alpha = \sum_I \alpha_I(\sigma, \varsigma) d\sigma^I \wedge d\varsigma^{1\dots r}$,

$$\begin{aligned}
\int_N \beta \wedge \int_{\pi_{\mathcal{N}}} \alpha &= \sum_I \int_N \beta \wedge d\sigma^I \int_{\varsigma \in \mathcal{N}_\sigma} i_I^{(\sigma)} \alpha|_{(\sigma, \varsigma)} = \sum_I \int_N \beta \wedge d\sigma^I \int_{\varsigma \in \mathcal{N}_\sigma} \alpha_I d\varsigma^{1\dots r} \\
&= \int_{\mathcal{N}} \pi_{\mathcal{N}}^*(\beta) \wedge \alpha
\end{aligned}$$

Hence (3). Conversely if (3) holds for $\alpha = \sum_I \alpha_I(\sigma, \varsigma) d\sigma^I \wedge d\varsigma^{1\dots r}$ then from (A1)

$$\int_N \beta \wedge \int_{\pi_{\mathcal{N}}} \alpha = \int_{\mathcal{N}} \pi_{\mathcal{N}}^*(\beta) \wedge \alpha = \sum_I \int_N \beta \wedge d\sigma^I \int_{\varsigma \in \mathcal{N}_\sigma} i_I^{(\sigma)} \alpha|_{(\sigma, \varsigma)}$$

Since this is true for all β then (4) holds.

If α does not contain the factor $\varsigma^{1\dots r}$ i.e. $\alpha = \alpha_{IK}(\sigma, \varsigma) d\sigma^I \wedge d\varsigma^K$ where $K \neq \{1, \dots, r\}$ then the right hand side of (3) becomes

$$\int_{\mathcal{N}} \pi_{\mathcal{N}}^*(\beta) \wedge \alpha = \sum_J \int_{\mathcal{N}} \beta_J \alpha_{IK}(\sigma, \varsigma) d\sigma^J \wedge d\sigma^I \wedge d\varsigma^K = 0$$

and the right hand side of (4) becomes

$$\sum_I d\sigma^I \int_{\varsigma \in \mathcal{N}_\sigma} \alpha_{IK}(\sigma, \varsigma) d\varsigma^K = 0$$

Thus by linearity (3) and (4) are equivalent for all α . □

Lemma 3. *Verification of equation (5):*

$$\left(d \int_{\pi_N} \alpha \right) \Big|_{\sigma} = \left(\int_{\pi_N} d\alpha \right) \Big|_{\sigma}$$

Proof. Let $\deg(\alpha) = p + r$, $\deg(\beta) = n - p - 1$ and ∂N and $\partial \mathcal{N}$ be the boundaries of N and \mathcal{N} . Since $\sigma \notin \partial N$ one may choose β to have support away from ∂N thus

$$\int_{\partial N} \beta \wedge \left(\int_{\pi_N} \alpha \right) = 0$$

and since α has support away from $\partial \mathcal{N}$ then

$$\int_{\partial \mathcal{N}} \pi_{\mathcal{N}}^* \beta \wedge \alpha = 0$$

It follows that

$$\begin{aligned} \int_N \beta \wedge \left(\int_{\pi_N} d\alpha \right) &= \int_N \pi_{\mathcal{N}}^*(\beta) \wedge d\alpha \\ &= (-1)^{n-p-1} \int_N d(\pi_{\mathcal{N}}^*(\beta) \wedge \alpha) + (-1)^{n-p} \int_N d\pi_{\mathcal{N}}^*(\beta) \wedge \alpha \\ &= (-1)^{n-p-1} \int_{\partial \mathcal{N}} \pi_{\mathcal{N}}^*(\beta) \wedge \alpha + (-1)^{n-p} \int_N \pi_{\mathcal{N}}^*(d\beta) \wedge \alpha \\ &= (-1)^{n-p} \int_N d\beta \wedge \left(\int_{\pi_N} \alpha \right) \\ &= (-1)^{n-p} \int_N d\left(\beta \wedge \left(\int_{\pi_N} \alpha \right) \right) + \int_N \beta \wedge d\left(\int_{\pi_N} \alpha \right) \\ &= (-1)^{n-p} \int_{\partial N} \beta \wedge \left(\int_{\pi_N} \alpha \right) + \int_N \beta \wedge d\left(\int_{\pi_N} \alpha \right) \\ &= \int_N \beta \wedge d\left(\int_{\pi_N} \alpha \right) \end{aligned}$$

□

Lemma 4. *Proof of*

$$\int_{p_X} \star_X \alpha = \star \int_{p_X} \alpha \quad (\text{A2})$$

Proof. The only non-trivial $\alpha \in \Gamma\Lambda(M_X \times M_Y)$ in (A2) can be written $\alpha = \alpha_I dx^I \wedge dy^{0123}$.

Then

$$\int_{p_X} \star_X (\alpha_I dx^I \wedge dy^{0123}) = \int_{p_X} \alpha_I (\star dx^I) \wedge dy^{0123} = \star dx^I \int_{M_X} \alpha_I dy^{0123} = \star \int_{p_X} \alpha_I dx^I \wedge dy^{0123}$$

□

Lemma 5. Π is causal on M_X^+ if and only if

- Z is causal on M_X^+ ,
 - $(d_Y \chi)|_{(x,y)} = 0$ for all $(x,y) \in M_X^+ \times M_Y^+$ such that $y \notin J^-(x)$ and
 - $\iota_{\Sigma_{M_Y}}^*(\chi)|_{(x,y)} = 0$ for all $(x,y) \in M_X^+ \times \Sigma_{M_Y}$ such that $y \notin J^-(x)$, where $\iota_{\Sigma_{M_Y}} : M_X^+ \times \Sigma_{M_Y} \hookrightarrow M_X^+ \times M_Y^+$ is the natural embedding.
- (A3)

Proof. If $\hat{\iota}_{\Sigma_{M_Y}} : \Sigma_{M_Y} \hookrightarrow M_Y^+$ is the natural embedding then $i_{ab}^{(x)} \iota_{\Sigma_{M_Y}}^* \chi|_{(x,y)} = \hat{\iota}_{\Sigma_{M_Y}}^* i_{ab}^{(x)} \chi|_{(x,y)}$, and

$$\begin{aligned}
& \int_{y \in M_Y^+} i_{ab}^{(x)} (\chi| \wedge p_Y^* (dA|_y)) \\
&= \int_{y \in M_Y^+} i_{ab}^{(x)} (\chi \wedge d_Y (p_Y^* A))|_{(x,y)} \\
&= \int_{y \in M_Y^+} i_{ab}^{(x)} d_Y (\chi \wedge p_Y^* A|_y)|_{(x,y)} - \int_{y \in M_Y^+} i_{ab}^{(x)} (d_Y \chi \wedge p_Y^* A|_y)|_{(x,y)} \\
&= \int_{y \in M_Y^+} d_Y (i_{ab}^{(x)} \chi \wedge p_Y^* A|_y)|_{(x,y)} - \int_{y \in M_Y^+} i_{ab}^{(x)} (d_Y \chi \wedge p_Y^* A|_y)|_{(x,y)} \\
&= \int_{y \in \Sigma_{M_Y}} \hat{\iota}_{\Sigma_{M_Y}}^* i_{ab}^{(x)} (\chi \wedge p_Y^* A|_y)|_{(x,y)} - \int_{y \in M_Y^+} i_{ab}^{(x)} (d_Y \chi \wedge p_Y^* A|_y)|_{(x,y)} \\
&= \int_{y \in \Sigma_{M_Y} \setminus J^-(x)} i_{ab}^{(x)} \iota_{\Sigma_{M_Y}}^* (\chi \wedge p_Y^* A|_y)|_{(x,y)} + \int_{y \in \Sigma_{M_Y} \cap J^-(x)} i_{ab}^{(x)} \iota_{\Sigma_{M_Y}}^* (\chi \wedge p_Y^* A|_y)|_{(x,y)} \\
&\quad - \int_{y \in M_Y^+ \setminus J^-(x)} i_{ab}^{(x)} (d_Y \chi \wedge p_Y^* A|_y)|_{(x,y)} - \int_{y \in M_Y^+ \cap J^-(x)} i_{ab}^{(x)} (d_Y \chi \wedge p_Y^* A|_y)|_{(x,y)}
\end{aligned}$$
(A4)

First one argues that (A3) implies that Π is causal on M_X^+ . Given $x \in M_X^+$ and $F_1, F_2 \in \Gamma \Lambda^2 M_Y^+$ such that $F_1|_y = F_2|_y = 0$ for $y \in J^-(x)$, set $F = F_1 - F_2$ so that $F = 0$ on $J^-(x)$. Since M_Y^+ is topologically trivial F is exact, $F = d\hat{A}$, and hence $d\hat{A} = 0$ on $J^-(x)$. Then since $J^-(x)$ is topologically trivial there exists $f \in \Gamma \Lambda^0 M_Y^+$ such that $\hat{A} = df$ on $J^-(x)$. Thus one can choose a gauge $A = \hat{A} - df$ so that $A = 0$ on $J^-(x)$. Given ζ such that $\zeta|_y = 0$ for $y \in J^-(x) \cap \Sigma_{M_Y}$ then $Z[\zeta]|_x = 0$ since Z is causal. Thus from (A4)

$$\begin{aligned}
\Pi[F, \zeta]_{ab}(x) &= \int_{y \in M_Y^+} i_{ab}^{(x)} (\chi| \wedge p_Y^* (dA|_y)) \\
&= \int_{y \in \Sigma_{M_Y} \setminus J^-(x)} i_{ab}^{(x)} \iota_{\Sigma_{M_Y}}^* (\chi \wedge p_Y^* A|_y)|_{(x,y)} + \int_{y \in \Sigma_{M_Y} \cap J^-(x)} i_{ab}^{(x)} \iota_{\Sigma_{M_Y}}^* (\chi \wedge p_Y^* A|_y)|_{(x,y)} \\
&\quad - \int_{y \in M_Y^+ \setminus J^-(x)} i_{ab}^{(x)} (d_Y \chi \wedge p_Y^* A|_y)|_{(x,y)} - \int_{y \in M_Y^+ \cap J^-(x)} i_{ab}^{(x)} (d_Y \chi \wedge p_Y^* A|_y)|_{(x,y)} = 0
\end{aligned}$$

since $\iota_{\Sigma_{M_Y}}^* \chi|_{(x,y)} = 0$ for $y \in \Sigma_{M_Y} \setminus J^-(x)$, $A|_y = 0$ for $y \in J^-(x)$ and $d_Y \chi = 0$ for $y \in M_Y^+ \setminus J^-(x)$.

Conversely if Π is causal on M_X^+ then setting $F = 0$ in (6) shows that Z must be causal on M_X^+ . Then setting $\zeta = 0$ then for all A such that $A = 0$ on $J^-(x)$ (A4) yields

$$\begin{aligned} 0 &= \Pi[F, \zeta]_{ab}(x) \\ &= \int_{y \in \Sigma_{M_Y} \setminus J^-(x)} i_{ab}^{(x)} \iota_{\Sigma_{M_Y}}^* (\chi_{(x,y)} \wedge p_Y^* A|_y)|_{(x,y)} - \int_{y \in M_Y^+ \setminus J^-(x)} i_{ab}^{(x)} (d_Y \chi_{(x,y)} \wedge p_Y^* A|_y)|_{(x,y)} \end{aligned} \quad (\text{A5})$$

The 4-dimensional domain $M_Y^+ \setminus J^-(x)$ denotes points outside the backward lightcone of x , while the 3-dimensional domain $\Sigma_{M_Y} \setminus J^-(x)$ denotes the points on Σ_{M_Y} that are not causally connected to x . Choosing such an A to have support about a small neighbourhood of $y \in M_Y^+ \setminus J^-(x) \setminus \Sigma_{M_Y}$ results in the first term of (A5) being zero and thus $(d_Y \chi)|_{(x,y)} = 0$. Likewise setting A to have support about a small neighbourhood of $y \in \Sigma_{M_Y} \setminus J^-(x)$ implies $\iota_{\Sigma_{M_Y}}^* (\chi)|_{(x,y)} = 0$.

□

One can now prove **lemma 1** in section III B.

Proof of lemma 1. Given $\sigma \in N$, with V non vanishing there exists a coordinate system $(\sigma^1, \dots, \sigma^n)$ on N adapted to V so that $V = \frac{\partial}{\partial \sigma^1}$ and the image of the curve $\gamma_\sigma : [\tau_0(\sigma), 0] \rightarrow N$ is contained in the coordinate patch. Write $\beta = \beta_I d\sigma^I$ then since $i_V \beta = 0$ the sum is over $I \in \{2, \dots, n\}$. With σ^1 distinguished write $\beta_I(\sigma) = \beta_I(\sigma^1, \underline{\sigma})$ where $\underline{\sigma} = (\sigma^2, \dots, \sigma^n)$. Also since $i_V \beta = 0$, $\beta|_{(\sigma^1, \underline{\sigma})} = \beta_I(\sigma^1, \underline{\sigma}) d\underline{\sigma}^I$. Likewise since $i_V \zeta = 0$ one has $\zeta|_{\sigma_0} = \zeta_I(\sigma_0) d\underline{\sigma}^I$.

Solving for the integral curves of V gives $\phi_N(\tau, \sigma^1, \underline{\sigma}) = (\tau + \sigma^1, \underline{\sigma})$

$$\phi_N^*(\beta)|_{(\tau, \sigma^1, \underline{\sigma})} = \beta_I(\tau + \sigma^1, \underline{\sigma}) d\underline{\sigma}^I$$

and one may write $\tau_0(\sigma^1, \underline{\sigma}) = \tau_0(\underline{\sigma}) - \sigma^1$, giving

$$\varphi_N(\zeta)|_{(\sigma^1, \underline{\sigma})} = \phi_{N\tau_0(\sigma^1, \underline{\sigma})}^*(\zeta|_{\tau_0(\underline{\sigma})}) = \zeta_I(\tau_0(\underline{\sigma}), \underline{\sigma}) d\underline{\sigma}^I$$

Thus

$$\begin{aligned} \xi|_{(\sigma^1, \underline{\sigma})} &= \int_{\varpi_N} \phi_N^*(\beta) \wedge d\tau + \varphi_N(\zeta) \\ &= \left(\int_{\tau=\tau_0(\underline{\sigma})-\sigma^1}^0 \beta_I(\sigma^1 + \tau, \underline{\sigma}) d\tau + \zeta_I(\tau_0(\underline{\sigma}), \underline{\sigma}) \right) d\underline{\sigma}^I \end{aligned}$$

Hence $i_V \xi = 0$ and one may write $\xi|_{(\sigma^1, \underline{\sigma})} = \xi_I(\sigma^1, \underline{\sigma}) d\underline{\sigma}^I$. Now

$$\begin{aligned}\xi_I(\sigma^1, \underline{\sigma}) &= \int_{\tau=\tau_0(\underline{\sigma})-\sigma^1}^0 \beta_I(\sigma^1 + \tau, \underline{\sigma}) d\tau + \zeta_I(\tau_0(\underline{\sigma}), \underline{\sigma}) \\ &= \int_{\tau=\tau_0(\underline{\sigma})}^{\sigma^1} \beta_I(\tau', \underline{\sigma}) d\tau' + \zeta_I(\tau_0(\underline{\sigma}), \underline{\sigma})\end{aligned}$$

where $\tau' = \tau + \sigma^1$ and

$$\begin{aligned}i_V d\xi|_{(\sigma^1, \underline{\sigma})} &= i_{\frac{\partial}{\partial \sigma^1}} d(\xi_I(\sigma^1, \underline{\sigma}) d\underline{\sigma}^I) = i_{\frac{\partial}{\partial \sigma^1}} (d\xi_I(\sigma^1, \underline{\sigma}) \wedge d\underline{\sigma}^I) = \frac{\partial \xi_I(\sigma^1, \underline{\sigma})}{\partial \sigma^1} d\underline{\sigma}^I \\ &= \frac{\partial}{\partial \sigma^1} \left(\int_{\tau=\tau_0(\underline{\sigma})}^{\sigma^1} \beta_I(\tau', \underline{\sigma}) d\tau' + \zeta_I(\tau_0(\underline{\sigma}), \underline{\sigma}) \right) d\underline{\sigma}^I = \beta_I(\sigma^1, \underline{\sigma}) d\underline{\sigma}^I = \beta|_{(\sigma^1, \underline{\sigma})}\end{aligned}$$

Since $\sigma^1 = 0$ on Σ_N

$$\xi|_{(0, \underline{\sigma})} = \xi_I(\tau_0(0, \underline{\sigma}), \underline{\sigma}) d\underline{\sigma}^I = \xi_I(\tau_0(\underline{\sigma}), \underline{\sigma}) d\underline{\sigma}^I = \zeta_I(\tau_0(\underline{\sigma}), \underline{\sigma}) d\underline{\sigma}^I = \zeta|_{(0, \underline{\sigma})}$$

i.e. $\xi|_{\Sigma_N} = \zeta$.

□

Lemma 6. *Proof that (66,67) implies (61) and that (66,69) implies (61).*

Proof. First (67) is equivalent to (69) since given $\gamma \in \Gamma \Lambda^2 M_Y^+$ one has $\widetilde{i_{(y,u)} \gamma} = u^a \gamma_{ab} g^{bc} \frac{\partial}{\partial y^c}$ and hence $\hat{W}^{[\alpha]}(\gamma) = \frac{q^{[\alpha]}}{m^{[\alpha]}} \mathcal{V}_{(y,u)}(\widetilde{i_{(y,u)} \gamma}) = \frac{q^{[\alpha]}}{m^{[\alpha]}} u^a \gamma_{ab} g^{b\nu} \frac{\partial}{\partial u^\nu}$. From (67) it follows that

$$\begin{aligned}\chi^{[\alpha]} \wedge p_Y^* \gamma &= \frac{1}{2} \frac{q^{[\alpha]^2}}{m^{[\alpha]}} \star_X dy^{cd} \wedge i_{abcd}^{(y)} \Psi^{[\alpha]*} \left(d\tau \wedge \varpi_Y^{[\alpha]*} (g^{\nu a} u^b i_\nu^{(u)} \theta_0^{[\alpha]}) \right) \wedge p_Y^* \gamma \\ &= \frac{q^{[\alpha]^2}}{m^{[\alpha]}} \star_X S i_{ab}^{(y)} \Psi^{[\alpha]*} \left(d\tau \wedge \varpi_Y^{[\alpha]*} (g^{\nu a} u^b i_\nu^{(u)} \theta_0^{[\alpha]}) \right) \wedge p_Y^* \gamma \\ &= -\frac{q^{[\alpha]^2}}{m^{[\alpha]}} \star_X S \Psi^{[\alpha]*} \left(d\tau \wedge \varpi_Y^{[\alpha]*} (g^{\nu a} u^b i_\nu^{(u)} \theta_0^{[\alpha]}) \right) \wedge i_{ab}^{(y)} p_Y^* \gamma \\ &= -\frac{q^{[\alpha]^2}}{m^{[\alpha]}} \star_X S \Psi^{[\alpha]*} \left(d\tau \wedge \varpi_Y^{[\alpha]*} (\gamma_{ab} g^{\nu a} u^b i_\nu^{(u)} \theta_0^{[\alpha]}) \right) \\ &= -q^{[\alpha]} \star_X S \Psi^{[\alpha]*} \left(d\tau \wedge \varpi_Y^{[\alpha]*} (i_{\hat{W}^{[\alpha]}(\gamma)} \theta_0^{[\alpha]}) \right)\end{aligned}$$

i.e. (69). That (69) implies (67) follows since the above argument is true for all γ .

To prove (61) note that the domains $\mathcal{N}_X^{[\alpha]}$ and $\mathcal{N}_Y^{[\alpha]}$ are related via the diffeomorphism

$$\Upsilon^{[\alpha]} : \mathcal{N}_Y^{[\alpha]} \rightarrow \mathcal{N}_X^{[\alpha]}, \quad \Upsilon^{[\alpha]}(\tau, y, u) = (-\tau, \dot{C}_{(y,u)}^{[\alpha]}(\tau)) \quad (\text{A6})$$

Thus $\Upsilon^{[\alpha]*}(d\tau) = -d\tau$ and setting $(x, v) = \dot{C}_{(y,u)}^{[\alpha]}(\tau)$ with $\tau > 0$ yields

$$\phi^{[\alpha]}(\Upsilon^{[\alpha]}(\tau, y, u)) = \phi^{[\alpha]}(-\tau, \dot{C}_{(y,u)}^{[\alpha]}(\tau)) = \phi^{[\alpha]}(-\tau, x, v) = (y, u) = \varpi_Y^{[\alpha]}(\tau, y, u)$$

so that $\varpi_Y^{[\alpha]} = \phi^{[\alpha]} \circ \Upsilon^{[\alpha]}$ and thus $\varpi_Y^{[\alpha]\star} = \Upsilon^{[\alpha]\star} \circ \phi^{[\alpha]\star}$. Now

$$\Upsilon^{[\alpha]\star} \left(d\tau \wedge \phi^{[\alpha]\star} (i_{\dot{W}^{[\alpha]}(F_1)} \theta_0^{[\alpha]}) \right) = \Upsilon^{[\alpha]\star} (d\tau) \wedge \Upsilon^{[\alpha]\star} \phi^{[\alpha]\star} (i_{\dot{W}^{[\alpha]}(F_1)} \theta_0^{[\alpha]}) = -d\tau \wedge \varpi_Y^{[\alpha]\star} (i_{\dot{W}^{[\alpha]}(F_1)} \theta_0^{[\alpha]})$$

hence

$$\chi^{[\alpha]} \wedge p_Y^\star F_1 = q^{[\alpha]} \star_X S \Psi^{[\alpha]\star} \Upsilon^{[\alpha]\star} \left(d\tau \wedge \phi^{[\alpha]\star} (i_{\dot{W}^{[\alpha]}(F_1)} \theta_0^{[\alpha]}) \right) \quad (\text{A7})$$

From (63)

$$p_X(\Phi^{[\alpha]}(\tau, y, u)) = p_X(C_{(y,u)}^{[\alpha]}(\tau), y) = C_{(y,u)}^{[\alpha]}(\tau)$$

and from (A6)

$$\pi_X(\varpi_X^{[\alpha]}(\Upsilon^{[\alpha]}(\tau, y, u))) = \pi_X(\varpi_X^{[\alpha]}(-\tau, \dot{C}_{(y,u)}^{[\alpha]}(\tau))) = \pi_X(\dot{C}_{(y,u)}^{[\alpha]}(\tau)) = C_{(y,u)}^{[\alpha]}(\tau)$$

Hence $p_X \circ \Phi^{[\alpha]} = \pi_X \circ \varpi_X^{[\alpha]} \circ \Upsilon^{[\alpha]}$ and so

$$\Phi^{[\alpha]\star} \circ p_X^\star = \Upsilon^{[\alpha]\star} \circ \varpi_X^{[\alpha]\star} \circ \pi_X^\star \quad (\text{A8})$$

From the definition of S one has

$$\int_{p_X} S\gamma = \int_{p_X} \gamma \quad (\text{A9})$$

for any $\gamma \in \Gamma\Lambda^8(M_X \times M_Y)$.

Since $\Psi^{[\alpha]} : \mathcal{D} \rightarrow \mathcal{D}'$ is a diffeomorphism then

$$\int_{\mathcal{D}} \Psi^{[\alpha]\star} \gamma = \int_{\mathcal{D}'} \gamma \quad (\text{A10})$$

for any $\gamma \in \Gamma\Lambda^8(\mathcal{D}')$. Likewise since $\Upsilon^{[\alpha]} : \mathcal{N}_Y^{[\alpha]} \rightarrow \mathcal{N}_X^{[\alpha]}$ is a diffeomorphism

$$\int_{\mathcal{N}_Y^{[\alpha]}} \Upsilon^{[\alpha]\star} \gamma = \int_{\mathcal{N}_X^{[\alpha]}} \gamma \quad (\text{A11})$$

for any $\gamma \in \Gamma\Lambda^8(\mathcal{N}_X^{[\alpha]})$.

For convenience set $\alpha^{[\alpha]} = d\tau \wedge \phi^{[\alpha]\star} (i_{\dot{W}^{[\alpha]}(F_1)} \theta_0^{[\alpha]}) \in \Gamma\Lambda^5 \mathcal{N}_X^{[\alpha]}$. For fixed x assume that F_1 has support in \mathcal{D}_x . Then one can choose $\beta \in \Gamma\Lambda^2 M_X$ so that $p_X^\star \beta \wedge p_Y^\star F_1$ has support inside \mathcal{D} . Thus from (A7)

$$\text{supp}(p_X^\star(\star\beta) \wedge \Psi^{[\alpha]\star} \Upsilon^{[\alpha]\star} \alpha^{[\alpha]}) = \text{supp}(p_X^\star \beta \wedge \chi^{[\alpha]} \wedge p_Y^\star F_1) \subset \mathcal{D} \quad (\text{A12})$$

Now

$$\begin{aligned}
\int_{M_X} \beta \wedge \oint_{p_X} \chi^{[\alpha]} \wedge p_Y^* F_1 &= \int_{M_X} \beta \wedge \oint_{p_X} q^{[\alpha]} \star_X S \Psi^{[\alpha]} \star \Upsilon^{[\alpha]} \star \alpha^{[\alpha]} && \text{from (A7)} \\
&= q^{[\alpha]} \int_{M_X} \beta \wedge \star \oint_{p_X} S \Psi^{[\alpha]} \star \Upsilon^{[\alpha]} \star \alpha^{[\alpha]} && \text{from (A2)} \\
&= q^{[\alpha]} \int_{M_X} \beta \wedge \star \oint_{p_X} \Psi^{[\alpha]} \star \Upsilon^{[\alpha]} \star \alpha^{[\alpha]} && \text{from (A9)} \\
&= -q^{[\alpha]} \int_{M_X} (\star \beta) \wedge \oint_{p_X} \Psi^{[\alpha]} \star \Upsilon^{[\alpha]} \star \alpha^{[\alpha]} \\
&= -q^{[\alpha]} \int_{M_X \times M_Y} p_X^* (\star \beta) \wedge \Psi^{[\alpha]} \star \Upsilon^{[\alpha]} \star \alpha^{[\alpha]} && \text{from (3)} \\
&= -q^{[\alpha]} \int_{\mathcal{D}} p_X^* (\star \beta) \wedge \Psi^{[\alpha]} \star \Upsilon^{[\alpha]} \star \alpha^{[\alpha]} && \text{from (A12)} \\
&= -q^{[\alpha]} \int_{\mathcal{D}} \Psi^{[\alpha]} \star \left(\Phi^{[\alpha]} \star p_X^* (\star \beta) \wedge \Upsilon^{[\alpha]} \star \alpha^{[\alpha]} \right) && \text{from (64)} \\
&= -q^{[\alpha]} \int_{\mathcal{D}'} \Phi^{[\alpha]} \star p_X^* (\star \beta) \wedge \Upsilon^{[\alpha]} \star \alpha^{[\alpha]} && \text{from (A10)} \\
&= -q^{[\alpha]} \int_{\mathcal{N}_Y^{[\alpha]}} \Phi^{[\alpha]} \star p_X^* (\star \beta) \wedge \Upsilon^{[\alpha]} \star \alpha^{[\alpha]} && \text{since } \mathcal{D}' \subset \mathcal{N}_Y^{[\alpha]} \\
&= -q^{[\alpha]} \int_{\mathcal{N}_Y^{[\alpha]}} \Upsilon^{[\alpha]} \star \varpi_X^{[\alpha]} \star \pi_X^* (\star \beta) \wedge \Upsilon^{[\alpha]} \star \alpha^{[\alpha]} && \text{from (A8)} \\
&= -q^{[\alpha]} \int_{\mathcal{N}_X^{[\alpha]}} \varpi_X^{[\alpha]} \star \pi_X^* (\star \beta) \wedge \alpha^{[\alpha]} && \text{from (A11)} \\
&= -q^{[\alpha]} \int_{\mathcal{E}_X} \pi_X^* (\star \beta) \wedge \oint_{\varpi_X^{[\alpha]}} \alpha^{[\alpha]} && \text{from (3)} \\
&= -q^{[\alpha]} \int_{M_X} (\star \beta) \wedge \oint_{\pi_X} \oint_{\varpi_X^{[\alpha]}} \alpha^{[\alpha]} && \text{from (3)} \\
&= q^{[\alpha]} \int_{M_X} \beta \wedge \star \oint_{\pi_X} \oint_{\varpi_X^{[\alpha]}} \alpha^{[\alpha]}
\end{aligned}$$

Summing over $[\alpha]$ gives

$$\int_{M_X} \beta \wedge \oint_{p_X} \chi \wedge p_Y^* F_1 = \sum_{[\alpha]} q^{[\alpha]} \int_{M_X} \beta \wedge \star \oint_{\pi_X} \oint_{\varpi_X^{[\alpha]}} \alpha^{[\alpha]}$$

Since this is true for all β with support in a neighbourhood of x then (61) holds at x . \square

Lemma 7. *The derivation of (72) from (67).*

Proof. The derivation of (72) from (67) follows by first writing the Liouville vector field (36) as

$$W_0^{[\alpha]} = u^a \frac{\partial}{\partial y^a} + H^\nu \frac{\partial}{\partial u^\nu} \quad \text{where} \quad H^\nu = -\Gamma^\nu_{ef} u^e u^f + \frac{q^{[\alpha]}}{m^{[\alpha]}} F_{0ef} g^{\nu e} u^f$$

Then setting $f^{[\alpha]}(y, u) = f_0^{[\alpha]}(y, u) + f_1^{[\alpha]}(y, u)$ it follows from (31) that

$$\begin{aligned}\theta_0^{[\alpha]} &= i_{W_0^{[\alpha]}}(f_0^{[\alpha]}\Omega) = f_0^{[\alpha]}i_{W_0^{[\alpha]}}\left(\frac{|\det g|}{u_0}dy^{0123} \wedge du^{123}\right) \\ &= f_0^{[\alpha]}\frac{|\det g|}{u_0}\left(u^c Y_c \wedge du^{123} + \frac{1}{2}H^\mu \epsilon_{\mu\nu\sigma} Y \wedge du^{\nu\sigma}\right)\end{aligned}$$

where $Y_a = i_{\frac{\partial}{\partial y^a}} dy^{0123}$ and $Y = dy^{0123}$. Consequently

$$g^{\mu a} u^b i_\mu^{(u)} \theta_0^{[\alpha]} = f_0^{[\alpha]} \frac{|\det g|}{u_0} g^{\mu a} u^b \left(-\frac{1}{2} u^c \epsilon_{\mu\nu\sigma} Y_c \wedge du^{\nu\sigma} - H^\nu \epsilon_{\mu\nu\sigma} Y \wedge du^\sigma \right)$$

and

$$-d\tau \wedge g^{\mu a} u^b i_\mu^{(u)} \theta_0^{[\alpha]} = f_0^{[\alpha]} \frac{|\det g|}{u_0} g^{\mu a} u^b \epsilon_{\mu\nu\sigma} \left(\frac{u^c}{2} d\tau \wedge Y_c \wedge du^{\nu\sigma} + H^\nu d\tau \wedge Y \wedge du^\sigma \right)$$

Under the maps $\varpi_Y^{[\alpha]}$ and $\hat{\psi}^{[\alpha]\star}$ one has

$$\varpi_Y^{[\alpha]\star}(dy^a) = dy^a, \quad \varpi_Y^{[\alpha]\star}(du^\mu) = du^\mu$$

and

$$\hat{\psi}^{[\alpha]\star}(dy^a) = dy^a, \quad \hat{\psi}^{[\alpha]\star}(du^\mu) = \frac{\partial u^\mu}{\partial x^a} dx^a + \frac{\partial u^\mu}{\partial y^a} dy^a, \quad \hat{\psi}^{[\alpha]\star}(d\tau) = \frac{\partial \tau}{\partial x^a} dx^a + \frac{\partial \tau}{\partial y^a} dy^a$$

So using the projector S given in (70) yields

$$\begin{aligned}-S\hat{\psi}^{[\alpha]\star}(d\tau \wedge g^{\nu a} u^b i_\nu^{(u)} \theta_0^{[\alpha]}) &= f_0^{[\alpha]} \frac{|\det g|}{u_0} g^{\mu a} u^b \epsilon_{\mu\nu\sigma} \left(\frac{u^c}{2} \frac{\partial \tau}{\partial y^c} \frac{\partial u^\nu}{\partial x^d} \frac{\partial u^\sigma}{\partial x^e} - \frac{u^c}{2} \frac{\partial \tau}{\partial x^d} \frac{\partial u^\nu}{\partial y^c} \frac{\partial u^\sigma}{\partial x^e} \right. \\ &\quad \left. + \frac{u^c}{2} \frac{\partial \tau}{\partial x^d} \frac{\partial u^\nu}{\partial x^e} \frac{\partial u^\sigma}{\partial y^c} + H^\nu \frac{\partial \tau}{\partial x^d} \frac{\partial u^\sigma}{\partial x^e} \right) Y \wedge dx^{de}\end{aligned}$$

Hence from (67)

$$\begin{aligned}\chi^{[\alpha]} &= -\frac{q^{[\alpha]2}}{m^{[\alpha]}} \star_X \left(i_{ab}^{(y)} S\hat{\psi}^{[\alpha]\star} \left(d\tau \wedge \varpi_Y^\star(g^{\nu a} u^b i_\nu^{(u)} \theta_0^{[\alpha]}) \right) \right) \\ &= \frac{q^{[\alpha]2}}{m^{[\alpha]}} \star_X i_{ab}^{(y)} \left(f_0^{[\alpha]} \frac{|\det g|}{u_0} g^{\mu a} u^b \epsilon_{\mu\nu\sigma} \left(\frac{u^c}{2} \frac{\partial \tau}{\partial y^c} \frac{\partial u^\nu}{\partial x^d} \frac{\partial u^\sigma}{\partial x^e} - \frac{u^c}{2} \frac{\partial \tau}{\partial x^d} \frac{\partial u^\nu}{\partial y^c} \frac{\partial u^\sigma}{\partial x^e} \right. \right. \\ &\quad \left. \left. + \frac{u^c}{2} \frac{\partial \tau}{\partial x^d} \frac{\partial u^\nu}{\partial x^e} \frac{\partial u^\sigma}{\partial y^c} + H^\nu \frac{\partial \tau}{\partial x^d} \frac{\partial u^\sigma}{\partial x^e} \right) Y \wedge dx^{de} \right) \\ &= -\star_X \left(f_0^{[\alpha]} \frac{|\det g|}{u_0} g^{\mu a} u^b \epsilon_{\mu\nu\sigma} \epsilon_{abfg} \left(\frac{u^c}{2} \frac{\partial \tau}{\partial y^c} \frac{\partial u^\nu}{\partial x^d} \frac{\partial u^\sigma}{\partial x^e} - \frac{u^c}{2} \frac{\partial \tau}{\partial x^d} \frac{\partial u^\nu}{\partial y^c} \frac{\partial u^\sigma}{\partial x^e} \right. \right. \\ &\quad \left. \left. + \frac{u^c}{2} \frac{\partial \tau}{\partial x^d} \frac{\partial u^\nu}{\partial x^e} \frac{\partial u^\sigma}{\partial y^c} + H^\nu \frac{\partial \tau}{\partial x^d} \frac{\partial u^\sigma}{\partial x^e} \right) dx^{de} \wedge dy^{fg} \right) \\ &= \frac{q^{[\alpha]2}}{m^{[\alpha]}} f_0^{[\alpha]} \frac{|\det g|^{3/2}}{2u_0} g^{\mu b} u^a \epsilon_{\mu\nu\sigma} \epsilon_{abfg} \epsilon^{dehi} \left(\frac{u^c}{2} \frac{\partial \tau}{\partial y^c} \frac{\partial u^\nu}{\partial x^d} \frac{\partial u^\sigma}{\partial x^e} - \frac{u^c}{2} \frac{\partial \tau}{\partial x^d} \frac{\partial u^\nu}{\partial y^c} \frac{\partial u^\sigma}{\partial x^e} \right. \\ &\quad \left. + \frac{u^c}{2} \frac{\partial \tau}{\partial x^d} \frac{\partial u^\nu}{\partial x^e} \frac{\partial u^\sigma}{\partial y^c} + H^\nu \frac{\partial \tau}{\partial x^d} \frac{\partial u^\sigma}{\partial x^e} \right) dx_{hi} \wedge dy^{fg}\end{aligned}$$

□